HW-5

As always, “map” is used for “morphism”.
Read the definitions of an exact, additive, and abelian categories from the notes handed out in class (downloadable from the website).

Definition 1. *Let $\mathscr{C}$ be an exact category. An* exact sequence *in $\mathscr{C}$ is a sequence of maps in $\mathscr{C}$*

$$\dots C^{p-1} \xrightarrow{d^{p-1}} C^p \xrightarrow{d^p} C^{p+1} \xrightarrow{d^{p+1}} \dots$$

such that $\mathbf{im}\, d^{p-1} = \ker d^p$*,* $p \in \mathbb{Z}$*.*

Definition 2. *If $\mathscr{C}$ is any category, its* opposite category*, $\mathscr{C}^\circ$ is the category whose objects are the objects of $\mathscr{C}$, and a morphism $A \to B$ in $\mathscr{C}^\circ$ is a morphism $B \to A$ in $\mathscr{C}$. A* contravariant *functor on $\mathscr{C}$ is nothing but a functor on $\mathscr{C}^\circ$.*

(1) Suppose $\mathscr{C}$ is an exact category. Show that $\mathscr{C}^\circ$ is also an exact category, where, if $\alpha\colon A \to B$ is a map in $\mathscr{C}^\circ$ corresponding to $\breve{\alpha}\colon B \to A$ in $\mathscr{C}$, then $\ker \alpha$ corresponds to $\operatorname{coker} \breve{\alpha}$, etc., etc.

(2) Let $\mathscr{C}$ be an exact category. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & R \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ 0 & \longrightarrow & P' & \longrightarrow & Q' & \longrightarrow & R' \end{array}$$

with exact rows.

(a) Show that $\ker \alpha \to \ker \beta$ is the kernel for $\ker \beta \to R$.

(b) Show that $0 \to \ker \alpha \to \ker \beta \to \ker \gamma$ is exact.

(c) What would be the “dual” statement to (2b)? In other words consider the statement (2b) in the exact category $\mathscr{C}^\circ$ and translate that statement back to $\mathscr{C}$, and tell me what the new statement is.

(3) Let

$$\begin{array}{ccccccc} G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & 0 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} & & \\ G' & \longrightarrow & H' & \longrightarrow & I' & \longrightarrow & 0 \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

be a commutative diagram in an exact category $\mathscr{C}$ such that the rows are exact, and the left column is exact.

(a) Show that $\operatorname{coker} h \to \operatorname{coker} i$ is an isomorphism. (Use the last part of the previous problem is necessary.)
(b) Show that $G' \to \mathbf{im}\, h \to \mathbf{im}\, i$ is exact. [Hint: Replace P, Q, R, P', Q', R' in 2 by $\ker(H' \to I')$, H', I', 0, $\operatorname{coker} h$, and $\operatorname{coker} i$ respectively, and use the result from (2b).]
(c) Use the above to show that $\ker h \to \ker i$ is surjective.

(4) Let $\mathscr{C}$ be an exact category. Suppose we have a commutative diagram with exact rows, with the column on the extreme left also exact:

$$\begin{array}{ccccccc}
P & \longrightarrow & Q & \longrightarrow & R & \longrightarrow & S \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\
P' & \longrightarrow & Q' & \longrightarrow & R' & \longrightarrow & S' \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

Show that the induced sequence

$$\ker \beta \to \ker \gamma \to \ker \delta$$

is also exact.

Using the above we will prove the **snake lemma** in class. This lemma says that if we have an exact commutative diagram in an exact category:

$$\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \downarrow \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

then there is an exact sequence:

$$\ker \beta \to \ker \gamma \to \ker \delta \to \operatorname{coker} \beta \to \operatorname{coker} \gamma \to \operatorname{coker} \delta$$

(5) Assume the snake lemma. Show that if we have a commutative diagram

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

with exact rows in an exact category such that α, β, δ, and ϵ are isomorphisms, then so is γ.