HW-IV

Throughout A is a commutative ring. For any A-module M, $\operatorname{End}(M)$ (or if we wish to emphasise the role of the underlying ring A, $\operatorname{End}_A M$) will denote the ring of endomorphisms of M.

In what follows, E will be a finitely generated free A-module of rank n. For an A-map $T \in \operatorname{End}(E)$, define $\det T \in A$ to be the unique element such that $\wedge^n T\colon \wedge^n E \to \wedge^n E$ is given by $x_1 \wedge \cdots \wedge x_n \mapsto \det T(x_1 \wedge \cdots \wedge x_n)$. The top exterior product of E, namely $\wedge^n E$ is denoted $\det E$.

The dual of E is the free module $E^\vee = \operatorname{Hom}_A(E, A)$.

For $T \in \operatorname{End}(E)$, $T^t \in \operatorname{End}(E^\vee)$ will denote the transpose of T, i.e., $(T^t(f))(x) = f(T(x))$ for $f \in E^\vee$ and $x \in E$.

(1) Show that $\det : \operatorname{End}(E) \to A$ is a ring homomorphism.

(2) Let $n = 2$. Show that

$$(a_{11}e_1 + a_{21}e_2) \wedge (a_{12}e_1 + a_{22}e_2) = (a_{11}a_{22} - a_{21}a_{12})e_a \wedge e_2.$$

(3) Let

$$0 \to E' \to E \to E''$$

be a short exact sequence of free modules (E as above), with $\operatorname{rank} E' = r$. Let $f_1, \dots, f_r \in E'$, $g_{r+1}, \dots, g_n \in E''$, and for $r \le i \le n$, let $f_i \in E$ be a preimage of $g_i \in E''$. Show that $f_1 \wedge \cdots \wedge f_n \in \wedge^n E$ does not depend on the choice of the preimages $f_{r+1}, \dots, f_n$. Prove that $f_1 \wedge \cdots \wedge f_r \otimes g_{r+1} \wedge \cdots \wedge g_n \mapsto f_1 \wedge \cdots \wedge f_n$ gives an isomorphism

$$\det E' \otimes_A \det E'' \xrightarrow{\ \sim\ } \det E.$$

In general, for any non-negative integer m, show

$$\bigoplus_{p+q=m} \wedge^p E' \otimes_A \wedge^q E'' \cong \wedge^m E$$

(4) Let $\varphi_r\colon \wedge^r E^\vee \to (\wedge^r E)^\vee$ be the map

$$(\varphi_r(f_1 \wedge \cdots \wedge f_r))(x_1 \wedge \cdots \wedge x_r) = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) f_i(x_{\sigma i}).$$

Show that φ_r is an isomorphism. Show also that the following diagram commutes

$$\begin{array}{ccc} \wedge^r E^\vee & \xrightarrow{\varphi_r} & (\wedge^r E)^\vee \\ {\scriptstyle \wedge^r(T^t)}\downarrow & & \downarrow{\scriptstyle (\wedge^r T)^t} \\ \wedge^r E^\vee & \xrightarrow[\varphi_r]{} & (\wedge^r E)^\vee \end{array}$$

for every $T \in \operatorname{End}(E)$.

Minor matters. Fix a basis $\{e_1, \dots, e_n\}$ of E. Let $T \in \mathrm{End}(E)$ and $v \in E$. For $1 \le j \le n$ let $T(j, v) \in \mathrm{End}(E)$ be the endomorphism defined by

$$T(j,v)(e_i) = \begin{cases} T(e_i) & i \neq j \\ v & i = j. \end{cases}$$

Let F be a free module of rank $n-1$ with basis $f_1, \dots, f_{n-1}$. For $1 \le i \le n$ define A-maps $\lambda_i \colon F \to E$ and $\mu_i \colon E \to F$ by

$$\lambda_i(f_j) = \begin{cases} e_j & 1 \le j < i \\ e_{j+1} & i \le j \le n-1 \end{cases}$$

and

$$\mu_i(e_j) = \begin{cases} f_j & 1 \le j < i \\ 0 & j = i \\ f_{j-1} & i < j \le n. \end{cases}$$

For $T \in \mathrm{End}(E)$ and $1 \le i, j \le n$ define $T_{ij} \in \mathrm{End}(F)$ by $T_{ij} = \mu_i \circ T \circ \lambda_j$. Note that T_{ij} is defined by the commutativity of the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{T} & E \\ {\scriptstyle\lambda_j}\uparrow & & \downarrow{\scriptstyle\mu_i} \\ F & \xrightarrow[T_{ij}]{} & F \end{array}$$

(5) Show that $\det(T) = \sum_{i=1}^{n} a_{ij} \det(T(j, e_i))$ where (a_{ij}) is the matrix of T with respect to the basis (e_i).
(6) Show that $\det(T(j, e_i)) = (-1)^{i+j} \det T_{ij}$. Deduce that $\det(T)$ is given by the Laplace expansion for the matrix of T with respect to (e_i) along a column.
(7) Show that

$$((\wedge^{n-1}T)(e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_n)) \wedge v = (-1)^{n-j} \det(T(e_j, v)) e_1 \wedge \cdots \wedge e_n.$$

(8) Show that if $(T(e_i))$ are linearly dependent, then $\det(T) = 0$.
(9) Show that if T is invertible then $\det(T)$ is invertible. Can you show the converse?