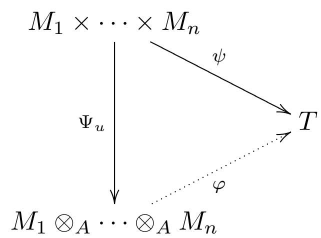
HW 3

Throughout, A is a commutative ring. Recall that if $M_1, \dots, M_n$ are A-modules, the tensor product $(M_1 \otimes_A \cdots \otimes_A M_n,\ \Psi_u)$ is an A-module $M_1 \otimes_A \cdots \otimes_A M_n$ together with an A-multilinear map $\Psi_u\colon M_1 \times \cdots \times M_n \to M_1 \otimes_A \cdots \otimes_A M_n$ such that if $\psi\colon M_1 \times \cdots \times M_n \to T$ is A-multilinear, there exists a unique map of A-modules $\varphi\colon M_1 \otimes_A \cdots \otimes_A M_n \to T$ such that $\varphi\colon \Psi_u = \psi$. In other words there is a unique way to fill the dotted arrow to make the diagram below commute:



In what follows, a "module" means an "A-module". Similarly, "multilinear", "bilinear" mean "A-multilinear" and "A-bilinear" respectively.

You are expected to use the universal property of tensor products to solve the problems below. You will never need the actual construction of the tensor product.

(1) Let M_1, M_2, M_3 be modules. Show that there are canonical isomorphisms

$$(M_1 \otimes_A M_2) \otimes_A M_3 \xrightarrow{\sim} M_1 \otimes_A M_2 \otimes_A M_3 \xrightarrow{\sim} M_1 \otimes_A (M_2 \otimes_A M_3).$$

(2) Let M, N, T be A-modules.
  (a) Suppose $\psi\colon M \times N \to T$ is bilinear. For each $m \in M$, let $\psi_m\colon N \to T$ be the map $n \mapsto \psi(m, n)$. Show that the map $\varphi\colon M \to \mathrm{Hom}_A(N, T)$ given by $m \mapsto \psi_m$ is A-linear.
  (b) Conversely, show that if $\varphi\colon M \to \mathrm{Hom}_A(N, T)$ is A-linear, then the map $\psi\colon M \times N \to T$ given by $(m, n) \mapsto \varphi(m)(n)$ is bilinear. Show also that $\varphi(m)$ is the map ψ_m of the part (a).
  (c) Show that we have a canonical isomorphism

$$\mathrm{Hom}_A(M, \mathrm{Hom}_A(N, T)) \xrightarrow{\sim} \mathrm{Hom}_A(M \otimes_A N, T).$$

(3) Let $M \in \mathrm{Mod}_A$. For an A-linear map $\varphi\colon N \to T$, let $M \otimes_A \varphi\colon M \otimes_A N \to M \otimes_A T$ be the map defined by $m \otimes n \mapsto m \otimes \varphi(n)$.
  (a) Show that $M \otimes_A \varphi$ is well defined.
  (b) Show that if

$$0 \to N' \to N \to N'' \to 0$$

is an exact sequence, then

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact.

Date: September 9, 2015.

(c) With the notations and hypotheses of part (b) show that

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

need not be exact.

(4) Let (M_λ) be a direct system of modules. Show that

$$(\varinjlim_{\lambda} M_\lambda) \otimes_A N = \varinjlim_{\lambda} (M_\lambda \otimes_A N).$$

(5) Let $M_1, \ldots, M_n$ be modules.

(a) Show that

$$(\bigoplus_{i=1}^{n} M_i) \otimes_A N = \bigoplus_{i=1}^{n} (M_i \otimes_A N).$$

(b) Show that

$$(\bigoplus_{\alpha \in I} M_\alpha) \otimes_A N = \bigoplus_{\alpha \in I} (M_\alpha \otimes_A N)$$

for an arbitrary direct sum $\bigoplus_{\alpha \in I} M_\alpha$.

(6) Let I be an ideal of A and M a module. Show that $M \otimes_A (A/I) = M/IM$.