HW 2

Throughout, A is a ring. Please consult Notes 2 (which has been uploaded) for definitions and orienting comments.

(1) Let $(\Lambda, \prec)$ be a directed set and $\Gamma \subset \Lambda$ a cofinal sub-directed set. Show that if $(M_\lambda)_{\lambda \in \Lambda}$ is a direct system, then there is an isomorphism

$$\mathrm{Hom}_\Lambda((M_\lambda)_{\lambda\in\Lambda}, T) \xrightarrow{\sim} \mathrm{Hom}_\Gamma((M_\gamma)_{\gamma\in\Gamma}, T)$$

for every module T.

(2) With the above notations show that

$$\varinjlim_{\lambda\in\Lambda} M_\lambda \xrightarrow{\sim} \varinjlim_{\gamma\in\Gamma} M_\gamma.$$

(3) Let k be a field, V a vector space over k, and $T_i \colon V \to V$, $i = 1, \ldots, n$ be n linear operators on V which *commute* with each other. on V. Let $A = k[X_1, \ldots, X_n]$, the polynomial ring in n-variables over k.

(a) Show that V is an A-module via the map $A \times V \to V$ given by $(f(X), v) \mapsto f(T)v$ for $f(X) \in A$ and $v \in V$.

(b) Suppose V is finite dimensional as a k-vector space, say $\dim_k V = d$. Show that $V_{\mathrm{tor}} = V$, where V_{tor} is the torsion module for V as an A-module (not as a k-module).

(4) Let A be commutative, I an ideal of A and M an A-module. Let $0 \underset{M}{:} I$ be the subset of M given by

$$0 \underset{M}{:} I = \{x \in M \mid ax = 0, \ \forall a \in I\}.$$

Show that $0 \underset{M}{:} I$ is a submodule of M. Show that $\mathrm{Hom}_A(A/I, M)$ is isomorphic to $0 \underset{M}{:} I$ as an A-module.

(5) Let A be commutative, and let I and M be as above. Define

$$\Gamma_I(M) = \{x \in M \mid I^n x = 0 \text{ for some } n > 0\}.$$

Show that $\Gamma_I(M)$ is an A-submodule. Show also that $(\mathrm{Hom}_A(A/I^n, M))_n$ is a direct system of A-modules, and that

$$\varinjlim_{n} \mathrm{Hom}_A(A/I^n, M) \xrightarrow{\sim} \Gamma_I(M).$$

(6) Suppose A is a commutative ring and $t \in A$ is an element. Let M be an A-module. Let $M_n = M$ for all $n \in \mathbb{N}$, and $\mu_{n,m} \colon M_m \to M_n$ the map $x \mapsto t^{n-m}x$ for $m < n$. Show that $(M_n, \mu_{n,m})$ is a direct system of A-modules. Show also (with M_t the localisation of M at t) that

$$\varinjlim_{n} M_n \xrightarrow{\sim} M_t.$$

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