HW 1

Throughout, A is a ring. Please consult Notes 1 (which has been uploaded) for definitions and orienting comments.

(1) Let $\eta\colon A \to \mathrm{End}_{\mathbb{Z}}(A)$ be the map defined in class, namely the map $x \mapsto x_L$ ($x \in A$), where x_L is 'multiplication on the left by x", i.e., $x_L(a) = xa$ for every $a \in A$. (Recall that we proved in class that η is *injective*.) Also, for $x \in A$, let x_R be "multiplication on the right by x", i.e., $x_R(a) = ax$ for $a \in A$. Let $A_L \subset \mathrm{End}_{\mathbb{Z}}(A)$ be the image of η, and similarly define $A_R = \{x_R \mid x \in A\}$. Show that $C(A_L) = A_R$ and $C(A_R) = A_L$.

(2) Let M_α, $\alpha \in I$, be submodules of a module M. Show that

$$\bigoplus_\alpha M_\alpha = \sum_\alpha M_\alpha$$

(i.e., the natural surjective map $\bigoplus_\alpha M_\alpha \twoheadrightarrow \sum_\alpha M_\alpha$ is an isomorphism) if and only if for every $\beta \in I$, the relationship $M_\beta \cap (\sum_{\alpha \neq \beta} M_\alpha) = 0$ holds.

(3) Let M be an A-module and $(s_\alpha \mid \alpha \in I)$ a linearly independent family in M.
  (a) For each α show that $A \cong As_\alpha$.
  (b) Let N be the submodule generated by $S = \{s_\alpha \mid \alpha \in I\}$. Show that $N = \bigoplus_{\alpha \in I} As_\alpha$.
  (c) Show that an A-module X is free if and only if $X \cong \bigoplus_{\gamma \in J} A$.

(4) Let $\mathfrak{a}$ be a left idea of A.
  (a) Show that $\mathfrak{a}(\bigoplus_{\gamma \in J} A) = \bigoplus_{\gamma \in J} \mathfrak{a}A$.
  (b) Let $M = \bigoplus_{\gamma \in J} A$. Show that

$$M/\mathfrak{a}M \cong \bigoplus_{\gamma \in J} A/\mathfrak{a}.$$

(5) Let A be a *commutative* ring. Let M be a free module. Show that any two bases of M have the same cardinality. (You may assume the result for vector spaces over a field.)

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