VECTOR FIELDS

This is a quick introduction to the subject, and we will only do as much as we need for the course. You will see all this in greater detail if you take a course on Differential Topology at some point.

1. **Derivations**

1.1. **Germs of smooth functions.** Let $\boldsymbol{a} \in \mathbf{R}^n$. Suppose Ω is an open neighbourhood of $\boldsymbol{a}$ in $\mathbf{R}^n$. An $\mathbf{R}$-linear map $D\colon \mathscr{C}^\infty(\Omega) \to \mathbf{R}$ is called a derivation on $\mathscr{C}^\infty(\Omega)$ at $\boldsymbol{a}$ if $D(fg) = f(\boldsymbol{a})Dg + g(\boldsymbol{a})Df$ for all $f, g \in \mathscr{C}^\infty(\Omega)$. For example, the operators

$$\mathbf{D}_{i,\boldsymbol{a}} = \frac{\partial}{\partial x_i}\Big|_{\boldsymbol{x}=\boldsymbol{a}}, \quad i = 1, \dots, n \tag{1.1.1}$$

are all derivations on $\mathscr{C}^\infty(\Omega)$ at $\boldsymbol{a}$. However, note that the displayed operators $\mathbf{D}_{i,\boldsymbol{a}}$ make sense on any open neighbourhood of $\boldsymbol{a}$. More importantly, $f, g \in \mathscr{C}^\infty(\Omega)$ are such that $f|_U = g|_U$ for some open neighbourhood U of $\boldsymbol{a}$, then $\mathbf{D}_{i,\boldsymbol{a}}(f) = \mathbf{D}_{i,\boldsymbol{a}}(g)$ for $i = 1, \dots, n$. Our goal is to mimic these properties of the $\mathbf{D}_{i,\boldsymbol{a}}$. To that end, define an equivalence relation $\overset{\boldsymbol{a}}{\sim}$ on the set of pairs (U, f) with U an open neighbourhood of $\boldsymbol{a}$ in $\mathbf{R}^n$ and $f\colon U \to \mathbf{R}$ a $\mathscr{C}^\infty$ function by the rule that $(U, f) \overset{\boldsymbol{a}}{\sim} (V, g)$ if there exists an set W, open in $\mathbf{R}^n$, with $\boldsymbol{a} \in W \subset U \cap V$, such that $f|_W = g|_W$. Note that writing the first member of the pair (U, f) is superfluous since a function comes with a domain. Nevertheless, since we sometimes need flexibility about the domain of a function, it is useful to write out pairs the way we have. That said, it is also useful not to so write. And we will do both, so long as the context is clear. As an aside, we write $\mathscr{D}\mathrm{om}(f)$ for the domain of a function f, in the event the domain has not been independently identified by us.

Definition 1.1.2. Let f be a $\mathscr{C}^\infty$ function defined in an open neighbourhood of $\boldsymbol{a}$. The *germ of f at $\boldsymbol{a}$* is the equivalence class of f under $\overset{\boldsymbol{a}}{\sim}$.

We will denote the set of germs (of $\mathscr{C}^\infty$ functions) at $\boldsymbol{a}$ by the symbol $\mathscr{C}^\infty_{\boldsymbol{a}}$. Note that $\mathscr{C}^\infty_{\boldsymbol{a}}$ is an $\mathbf{R}$-algebra. In greater detail, if $f_{\boldsymbol{a}}$ and $g_{\boldsymbol{a}}$ are two germs at $\boldsymbol{a}$, say $f_{\boldsymbol{a}}$ represented by f and $g_{\boldsymbol{a}}$ by g, then on $\mathscr{D}\mathrm{om}(f) \cap \mathscr{D}\mathrm{om}(g)$ we can multiply f and g. It is easy to see that the germ of fg at $\boldsymbol{a}$ does not depend on the representatives f and g of $f_{\boldsymbol{a}}$ and $g_{\boldsymbol{a}}$ respectively. We denote this germ of fg by $f_{\boldsymbol{a}}g_{\boldsymbol{a}}$. This defines a product on $\mathscr{C}^\infty_{\boldsymbol{a}}$ which is makes it into an $\mathbf{R}$-algebra. This fact is easy to verify. Note also that if $f_{\boldsymbol{a}} \in \mathscr{C}^\infty_{\boldsymbol{a}}$, then the value of $f_{\boldsymbol{a}}$ at $\boldsymbol{a}$ is well defined. In other words, if (U, f) and (V, g) both represent $f_{\boldsymbol{a}}$, then $f(\boldsymbol{a}) = g(\boldsymbol{a})$. We denote the value of $f_{\boldsymbol{a}}$ at $\boldsymbol{a}$ by $f_{\boldsymbol{a}}(\boldsymbol{a})$. For the algebraically minded:

$$\mathscr{C}^\infty_{\boldsymbol{a}} = \varinjlim_{U \ni \boldsymbol{a}} \mathscr{C}^\infty(U).$$

This fact is not important for this course, and feel free to ignore it.

Note that if we have $\boldsymbol{a} \in V \subset U$, with U and V open sets in $\mathbf{R}^n$, then the germ of (U, f) is equal to the germ of $(V, f|_V)$ for $f \in \mathscr{C}^\infty(U)$.

We will often use the symbols f and g for germs at $\boldsymbol{a}$ as well as for functions on subsets of $\mathbf{R}^n$. What we mean will be clear from the context.

Definition 1.1.3. An $\mathbf{R}$-linear map $D\colon \mathscr{C}^\infty_{\boldsymbol{a}} \to \mathbf{R}$ is said to be a *derivation at* $\boldsymbol{a}$ if $D(fg) = f(a)Dg + g(a)Df$ for all $f, g \in \mathscr{C}^\infty_{\boldsymbol{a}}$. The set of derivations at $\boldsymbol{a}$ is denoted $\mathscr{D}\mathrm{er}_{\boldsymbol{a}}$.

1.1.4. For D as above we often say *D is a derivation on $\mathscr{C}^\infty_a$*, or, when we wish to be precise, *D is an $\mathbf{R}$-valued derivation on $\mathscr{C}^\infty_{\boldsymbol{a}}$*. It is clear that the set $\mathscr{D}\mathrm{er}_{\boldsymbol{a}}$ of $\mathbf{R}$-valued derivations on $\mathscr{C}^\infty_{\boldsymbol{a}}$ is a vector space over $\mathbf{R}$. We will see in Theorem 1.1.9 below that it is in fact n-dimensional, with a basis consisting of the standard partial derivative operators $\mathbf{D}_{i,\boldsymbol{a}}$ at $\boldsymbol{a}$.

Example 1.1.5. The derivations $\mathbf{D}_{i,\boldsymbol{a}}$ defined on $\mathscr{C}^\infty(\Omega)$ (see (1.1.1))are really derivations on $\mathscr{C}^\infty_{\boldsymbol{a}}$ from the comments we made right after introducing them. Which way we regard them will be clear from the context.

Lemma 1.1.6. *Fix $\boldsymbol{a} \in \mathbf{R}^n$ and regard elements of $\mathbf{R}$ as elements of $\mathscr{C}^\infty_{\boldsymbol{a}}$ as well as of $\mathscr{C}^\infty(\Omega)$ for any open set Ω in $\mathbf{R}^n$.*

(a) *Let $D\colon \mathscr{C}^\infty_{\boldsymbol{a}} \to \mathbf{R}$ be a derivation. Then $D(c) = 0$ for $c \in \mathbf{R}$.*
(b) *Let Ω be an open set in $\mathbf{R}^n$ containing $\boldsymbol{a}$ and D a derivation on $\mathscr{C}^\infty(\Omega)$ at $\boldsymbol{a}$. Then $D(c) = 0$ for $c \in \mathbf{R}$.*

Proof. Let $D\colon \mathscr{C}^\infty_{\boldsymbol{a}} \to \mathbf{R}$ be a derivation. Since $1 = 1^2$, we have $D(1) = D(1^2) = 2D(1)$, whence $D(1) = 0$. Now if $c \in \mathbf{R}$, then $D(c) = D(c \cdot 1) = cD(1) = 0$. This proves (a). Part (b) is proved in exactly the same manner. ☐

Next suppose D is a derivation on $\mathscr{C}^\infty_{\boldsymbol{a}}$. Suppose f is a $\mathscr{C}^\infty$ function in an open neighbourhood of $\boldsymbol{a}$. We may assume this neighbourhood is a ball centred at $\boldsymbol{a} = (a_1, \dots, a_n)$. Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be a point in this neighbourhood, set $\boldsymbol{x} - \boldsymbol{a} = \boldsymbol{h} = (h_1, \dots, h_n)$. Set $g(t) = f(\boldsymbol{a} + t\boldsymbol{h})$ for $0 \le t \le 1$. Then, by the chain rule, $\dot{g}(t) = \sum_{i=1}^n h_i(\mathbf{D}_{i,\boldsymbol{a}+t\boldsymbol{h}} f) = \sum_{i=1}^n (x_i - a_i)(\mathbf{D}_{i,\boldsymbol{a}+t\boldsymbol{h}} f)$, whence

$$f(\boldsymbol{x}) = f(\boldsymbol{a}) + \int_0^1 \dot{g}(t)dt = f(\boldsymbol{a}) + \sum_{i=1}^n (x_i - a_i) \int_0^1 (\mathbf{D}_{i,\boldsymbol{a}+t\boldsymbol{h}} f)dt.$$

Since $\boldsymbol{h}$ is a function of $\boldsymbol{x}$, with $\boldsymbol{h} = \boldsymbol{x} - \boldsymbol{a}$, the above can be regarded as an expression for f as a function of $\boldsymbol{x}$. Hence we can apply D to it (or, more precisely, to its germ at $\boldsymbol{a}$). Applying D and noting that $\boldsymbol{h}(\boldsymbol{a}) = \mathbf{0}$ and $x_i(\boldsymbol{a}) - a_i = 0$ for all i, we get:

$$\begin{aligned} Df &= \sum_{i=1}^n \Big\{ D(x_i - a_i) \int_0^1 (\mathbf{D}_{i,\boldsymbol{a}+t\boldsymbol{h}(\boldsymbol{a})} f)dt + (x_i(\boldsymbol{a}) - a_i) D\Big(\int_0^1 (\mathbf{D}_{i,\boldsymbol{a}+t\boldsymbol{h}(\boldsymbol{x})} f)dt \Big) \Big\} \\ &= \sum_{i=1}^n D(x_i)(\mathbf{D}_{i,\boldsymbol{a}} f) \end{aligned}$$

For the last equality we are using the fact that $\boldsymbol{a} + t\boldsymbol{h}(\boldsymbol{a}) = \boldsymbol{a}$ and hence $\int_0^1 (\mathbf{D}_{i,\boldsymbol{a}+t\boldsymbol{h}(\boldsymbol{a})} f)dt = (\mathbf{D}_{i,\boldsymbol{a}} f) \int_0^1 dt = \mathbf{D}_{i,\boldsymbol{a}} f$. In other words if $\mu_i = D(x_i)$ then

$$D = \sum_{i=1}^n \mu_i \mathbf{D}_{i,\boldsymbol{a}}. \tag{1.1.7}$$

This is often written more evocatively as

$$D = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i}\Big|_{\boldsymbol{a}}. \tag{1.1.8}$$

Theorem 1.1.9. *For $\boldsymbol{a} \in \mathbf{R}^n$, $\dim_{\mathbf{R}} \mathscr{D}\mathrm{er}_{\boldsymbol{a}} = n$ and $\mathbf{D}_{i,\boldsymbol{a}}$, $i = 1, \ldots, n$, form a basis for $\mathscr{D}\mathrm{er}_{\boldsymbol{a}}$.*

Proof. From (1.1.7) it is clear that the collection $\{\mathbf{D}_{i,\boldsymbol{a}}\}_{i=1}^n$ spans $\mathscr{D}\mathrm{er}_{\boldsymbol{a}}$. If $\pi_j \colon \mathbf{R}^n \to \mathbf{R}$ is the projection to the j^{th} coordinate, and regarding π_j as a germ at $\boldsymbol{a}$, we have $\mathbf{D}_{i,\boldsymbol{a}}(\pi_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta symbol. Now suppose $c_1, \ldots, c_n$ are elements of $\mathbf{R}$ such that $\sum_i c_i \mathbf{D}_{i,\boldsymbol{a}} = 0$. Then $c_j = \left(\sum_i c_i \mathbf{D}_{i,\boldsymbol{a}}\right)(\pi_j) = 0$ for each $j \in \{1, \ldots, n\}$. Thus $\{\mathbf{D}_{i,\boldsymbol{a}}\}_{i=1}^n$ is a linearly independent collection. ☐

1.2. **Derivations and velocity vectors.** Let $\boldsymbol{a} \in V$, where V is an open set in $\mathbf{R}^n$. Suppose we have a $\mathscr{C}^1$ path $\boldsymbol{\gamma} \colon (\alpha, \beta) \to V$ and a point $t_\circ \in (\alpha, \beta)$ such that $\boldsymbol{\gamma}^{-1}(\boldsymbol{a}) = \{t_\circ\}$. Let $f_{\boldsymbol{a}} \in \mathscr{C}^\infty_{\boldsymbol{a}}$, and let (U, f) be a representative of $f_{\boldsymbol{a}}$, with (for simplicity)[1] $U \subset V$. Note that $t_\circ \in \boldsymbol{\gamma}^{-1}(U)$ and hence it makes sense to talk about $f \circ \boldsymbol{\gamma}$ on $\boldsymbol{\gamma}^{-1}(U)$. We define $D_{\boldsymbol{\gamma},\boldsymbol{a}}(f_{\boldsymbol{a}}) \in \mathbf{R}$ by the formula

$$D_{\boldsymbol{\gamma},\boldsymbol{a}}(f_{\boldsymbol{a}}) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \boldsymbol{\gamma})(t)\Big|_{t=t_\circ}.$$

It is clear that $D_{\boldsymbol{\gamma},\boldsymbol{a}}$ is well defined and that it is an $\mathbf{R}$-valued derivation on $\mathscr{C}^\infty_{\boldsymbol{a}}$.

$$D_{\boldsymbol{\gamma},\boldsymbol{a}}(f_{\boldsymbol{a}}) = f'(\boldsymbol{a})\dot{\boldsymbol{\gamma}}(t_\circ) = \langle \boldsymbol{\nabla}(f)(\boldsymbol{a}), \dot{\boldsymbol{\gamma}}(t_\circ)\rangle. \tag{1.2.1}$$

Note that $f'(\boldsymbol{a})$ depends only on the germ of (U, f) at $\boldsymbol{a}$. This means that $D_{\boldsymbol{\gamma},\boldsymbol{a}}$ depends only on the velocity vector $\dot{\gamma}(t_\circ)$.

Next, if γ_i is the i^{th} component of $\boldsymbol{\gamma}$, so that $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$, then (1.2.1) gives us $D_{\boldsymbol{\gamma},\boldsymbol{a}}(f_{\boldsymbol{a}}) = \sum_{i=1}^n \dot{\gamma}_i(t_\circ)\mathbf{D}_{i,\boldsymbol{a}}(f_{\boldsymbol{a}})$. Thus

$$D_{\boldsymbol{\gamma},\boldsymbol{a}} = \sum_{i=1}^{n} \dot{\gamma}_i(t_\circ)\mathbf{D}_{i,\boldsymbol{a}}. \tag{1.2.2}$$

From (1.2.1), we have a well defined linear map from the vector space $\mathrm{Vel}_{\boldsymbol{a}}$ of velocity vectors at $\boldsymbol{a}$ to $\mathscr{D}\mathrm{er}_{\boldsymbol{a}}$ and according to (1.2.2) this map is injective, since $\mathbf{D}_{i,\boldsymbol{a}}$, $i = 1, \ldots, n$, forms a basis for $\mathscr{D}\mathrm{er}_{\boldsymbol{a}}$. Since $\mathrm{Vel}_{\boldsymbol{a}} = \mathbf{R}^n$ we therefore see that the natural map $\mathrm{Vel}_{\boldsymbol{a}} \to \mathscr{D}\mathrm{er}_{\boldsymbol{a}}$ (given by $\boldsymbol{v} \mapsto \sum_{i=1}^n v_i \mathbf{D}_{i,\boldsymbol{a}}$) is an isomorphism.

1.3. **Change of coordinates.** Suppose $\boldsymbol{a}$ and V are as above. Let $\boldsymbol{\psi} \colon V \to W$ be diffeomorphism, with W an open subset of $\mathbf{R}^n$. Let $\boldsymbol{b} = \boldsymbol{\psi}(\boldsymbol{a})$. Now there is a one-to-one correspondence between smooth paths in V passing through $\boldsymbol{a}$ and smooth paths in W passing through $\boldsymbol{b}$, the correspondence being $\boldsymbol{\gamma} \mapsto \boldsymbol{\psi} \circ \boldsymbol{\gamma}$. Moreover if $\boldsymbol{\gamma}^{-1}(\boldsymbol{a})$ is a singleton set, say $\{t_0\}$, then $(\boldsymbol{\psi} \circ \boldsymbol{\gamma})^{-1}(\boldsymbol{b}) = \{t_0\}$. This means there is a one-to-one correspondence between velocity vectors at $\boldsymbol{a}$ and those at $\boldsymbol{b}$, namely, if $\boldsymbol{\sigma} = \boldsymbol{\psi} \circ \boldsymbol{\gamma}$, then the correspondence is $\dot{\boldsymbol{\gamma}}(t_0) \mapsto \dot{\boldsymbol{\sigma}}(t_0)$. From the chain rule, we see that

$$\dot{\boldsymbol{\sigma}}(t_0) = \boldsymbol{\psi}'(\boldsymbol{a})\dot{\boldsymbol{\gamma}}(t_0). \tag{1.3.1}$$

[1] Replace U with $U \cap V$ if necessary.

We know from the discussion in §§1.2 above that there is an $\mathbf{R}$-linear isomorphism between velocity vectors at $\boldsymbol{a}$ (respectively $\boldsymbol{b}$) and $\mathscr{D}\mathrm{er}_{\boldsymbol{a}}$ (respectively $\mathscr{D}\mathrm{er}_{\boldsymbol{b}}$). This gives us, via (1.3.1) and (1.2.2), a linear isomorphism

$$\begin{aligned} \boldsymbol{\psi}_* = \boldsymbol{\psi}_*(\boldsymbol{a}) \colon \mathscr{D}\mathrm{er}_{\boldsymbol{a}} &\xrightarrow{\sim} \mathscr{D}\mathrm{er}_{\boldsymbol{b}} \\ D_{\boldsymbol{\gamma},\boldsymbol{a}} &\longmapsto D_{(\boldsymbol{\psi}\circ\boldsymbol{\gamma}),\boldsymbol{b}}. \end{aligned} \tag{1.3.2}$$

The map $\boldsymbol{\psi}_*$ can also be described as follows. Let $D \in \mathscr{D}\mathrm{er}_{\boldsymbol{a}}$. Let $g_{\boldsymbol{b}} \in \mathscr{C}^\infty_{\boldsymbol{b}}$, say $g_{\boldsymbol{b}}$ is represented by (U', g) (and as before, we can assume $U' \subset W$). Setting $f = g \circ \boldsymbol{\psi}$ on $U = \boldsymbol{\psi}^{-1}(U')$, we get a germ $f_{\boldsymbol{a}} \in \mathscr{C}^\infty_{\boldsymbol{a}}$ represented by (U, f). We can define $\boldsymbol{\psi}_*(D)(g_{\boldsymbol{b}})$ by the formula

$$(\boldsymbol{\psi}_*(D))(g_{\boldsymbol{b}}) = D(f_{\boldsymbol{a}}). \tag{1.3.3}$$

It is clear that $\boldsymbol{\psi}_*(D) \in \mathscr{D}\mathrm{er}_{\boldsymbol{b}}$ since it is really the avatar of D under the natural isomorphism of $\mathbf{R}$-algebras $\mathscr{C}^\infty_{\boldsymbol{b}} \xrightarrow{\sim} \mathscr{C}^\infty_{\boldsymbol{a}}$ given by (ignoring annoying trivialities like the distinction between germs and functions representing germs) $g \mapsto g \circ \boldsymbol{\psi}$. In other words

$$\boldsymbol{\psi}_* \colon \mathscr{D}\mathrm{er}_{\boldsymbol{a}} \xrightarrow{\sim} \mathscr{D}\mathrm{er}_{\boldsymbol{b}}$$

is the isomorphism such that for each $D \in \mathscr{D}\mathrm{er}_{\boldsymbol{a}}$ the following diagram commutes:

$$\begin{array}{ccc} \mathscr{C}^\infty_{\boldsymbol{b}} & \xrightarrow{\quad\sim\quad} & \mathscr{C}^\infty_{\boldsymbol{a}} \\ & {\scriptstyle \boldsymbol{\psi}_*(D)} \searrow \quad \swarrow {\scriptstyle D} & \\ & \mathbf{R} & \end{array}$$

Since the horizontal arrow on top is an $\mathbf{R}$-algebra isomorphism, it is clear $\boldsymbol{\psi}_*(D)$ is an $\mathbf{R}$-derivation, since D is an $\mathbf{R}$-derivation. It is easy to see that the descriptions of $\boldsymbol{\psi}_*$ in (1.3.2) and (1.3.3) agree. This is really the chain rule. In summary we have the following result.

Theorem 1.3.4. *In the above situation, the matrix of $\boldsymbol{\psi}_* \colon \mathscr{D}\mathrm{er}_{\boldsymbol{a}} \xrightarrow{\sim} \mathscr{D}\mathrm{er}_{\boldsymbol{b}}$ with respect to the bases $\mathbf{D}_{1,\boldsymbol{a}}, \dots, \mathbf{D}_{n,\boldsymbol{a}}$ and $\mathbf{D}_{1,\boldsymbol{b}}, \dots, \mathbf{D}_{n,\boldsymbol{b}}$ on $\mathscr{D}\mathrm{er}_{\boldsymbol{a}}$ and $\mathscr{D}\mathrm{er}_{\boldsymbol{b}}$ respectively is the Jacobian matrix $(J\boldsymbol{\psi})(\boldsymbol{a})$ of $\boldsymbol{\psi}$ at $\boldsymbol{a}$.*

Proof. This is a trivial consequence of (1.3.1), (1.2.2), and the definition of $\boldsymbol{\psi}_*$ in (1.3.2). □

1.3.5. The tangent space at $\boldsymbol{a}$. In view of Problem 3 of HW 4 in ANA3 as well as the statements before Problem 5 of HW 5 in ANA2, it is clear from our discussions above that $\mathscr{D}\mathrm{er}_{\boldsymbol{a}}$ can be identified with the tangent space of V at $\boldsymbol{a}$. Since V is "flat", this does not seem to be useful. But the ideas translate to differential manifolds where these ideas are extremely useful. For this reason, we do regard $\mathscr{D}\mathrm{er}_{\boldsymbol{a}}$ as $T_{\boldsymbol{a}}(V)$, the tangent space to V at $\boldsymbol{a}$.

2. Vector Fields

2.1. **The tangent bundle.** Let V be an open set in $\mathbf{R}^n$. For $\boldsymbol{a} \in V$, we will use the notation $T_{\boldsymbol{a}}$ (or $T_{\boldsymbol{a}}(V)$) for $\mathscr{D}\mathrm{er}_{\boldsymbol{a}}$, in view of our comments in §§§1.3.5 above. We also write

$$T(V) := \bigcup_{\boldsymbol{a}\in V} T_{\boldsymbol{a}}(V). \tag{2.1.1}$$

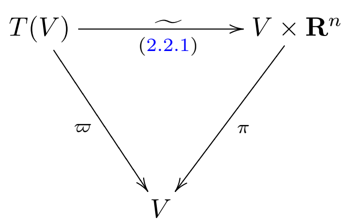
The union is clearly a disjoint union for $T_{\boldsymbol{a}} \cap T_{\boldsymbol{b}} = \emptyset$ if $\boldsymbol{a} \neq \boldsymbol{b}$. We therefore have a map $\varpi = \varpi_V : T(V) \to V$ given by $\varpi(D) = \boldsymbol{a}$ if $D \in T_{\boldsymbol{a}}$. We point out that $T_{\boldsymbol{a}} = \varpi^{-1}(\boldsymbol{a})$. *The* tangent bundle of V *is the pair* $(T(V), \varpi_V)$. We often omit the projection ϖ and simply call $T(V)$ the tangent bundle of V.

2.2. **Vector fields.** A map $\boldsymbol{X} : V \to \bigcup_{\boldsymbol{a}\in V} T_{\boldsymbol{a}}$ such that $\boldsymbol{X}(\boldsymbol{a}) \in T_{\boldsymbol{a}}$ for $\boldsymbol{a} \in V$ is called a *vector field on* V. A vector field $\boldsymbol{X}$ is clearly the same as a section of ϖ, i.e. $\boldsymbol{X}$ is a map such that $\varpi \circ \boldsymbol{X} = \mathbf{1}_V$, where $\mathbf{1}_V$ is the identity map on V. Now a special feature of open sets in $\mathbf{R}^n$ (versus open sets in "manifolds") is that we have a bijective map

(2.2.1)
$$\begin{aligned} T(V) &\xrightarrow{\sim} V \times \mathbf{R}^n \\ D &\longmapsto (\varpi(D), \boldsymbol{\mu}_D) \end{aligned}$$

where $\boldsymbol{\mu}_D = (\mu_1, \dots, \mu_n)$ is the unique vector in $\mathbf{R}^n$ such that $D = \sum_i \mu_i \mathbf{D}_{i,\boldsymbol{a}}$ (see (1.1.7)). In fact the diagram

(2.2.2)



commutes, where $\pi = \pi_V : V \times \mathbf{R}^n$ is the projection on to V. It is clear from the diagram above that sections of ϖ are in bijective correspondence with sections of π. Now, it is an easy set theoretic fact that if A and B are non-empty sets and $p : A \times B \to A$ is the projection to the first factor, then sections of p are completely determined by maps $f : A \to B$, via the "graph" of f. In other words f determines the section σ of p given by $a \mapsto (a, f(a))$ and a section σ of p determines a map $f : A \to B$ given by $f = q \circ \sigma$, where q is the projection $A \times B \to B$. These two processes are clearly inverses of each other.

Let $\boldsymbol{\sigma}_{\boldsymbol{X}} : V \to V \times \mathbf{R}^n$ be the section of π corresponding to the section $\boldsymbol{X}$ of ϖ. In view of what we just said, a vector field $\boldsymbol{X}$ on V corresponds to a map

(2.2.3)
$$\boldsymbol{\mu}_{\boldsymbol{X}} : V \to \mathbf{R}^n$$

(and vice-versa) in such a way that

(2.2.4)
$$\boldsymbol{\sigma}_{\boldsymbol{X}} = (\mathbf{1}_V, \boldsymbol{\mu}_{\boldsymbol{X}}).$$

More explicitly, if one unravels all the definitions involved, $\boldsymbol{\mu}_{\boldsymbol{X}}$ is described as follows. If $D_{\boldsymbol{a}} = \boldsymbol{X}(\boldsymbol{a})$ for a point $\boldsymbol{a}$ in V, and $\mathbf{D}_{\boldsymbol{a}} = \sum_{i=1}^n \mu_{\boldsymbol{X},i}(\boldsymbol{a}) \mathbf{D}_{i,\boldsymbol{a}}$ in the representation of $D_{\boldsymbol{a}}$ in the form (1.1.7), then $\boldsymbol{\mu}_{\boldsymbol{X}}(\boldsymbol{a}) = (\mu_{\boldsymbol{X},1}(\boldsymbol{a}), \dots, \mu_{\boldsymbol{X},n}(\boldsymbol{a}))$.

Definition 2.2.5. We say $\boldsymbol{X}$ is a *continuous vector field* if $\boldsymbol{\mu}_{\boldsymbol{X}}$ is continuous. We say it is a $\mathscr{C}^k$ *vector field* if $\boldsymbol{\mu}_{\boldsymbol{X}}$ is in $\mathscr{C}^k(V, \mathbf{R}^n)$. We say it is a *smooth vector field* if $\boldsymbol{\mu}_{\boldsymbol{X}} \in \mathscr{C}^\infty(V, \mathbf{R}^n)$.

2.3. **Change of co-ordinates for vector fields.** As in §§1.3, let $\boldsymbol{\psi}\colon V \to W$ be diffeomorphism, with W an open subset of $\mathbf{R}^n$. To reduce notational clutter, we write π for both projections $V \times \mathbf{R}^n \to V$ and $W \times \mathbf{R}^n \to \mathscr{W}$. Similarly we write ϖ for both $\varpi_V : T(V) \to V$ as well as for $\varpi_W : T(W) \to W$.

The linear isomorphisms $\boldsymbol{\psi}_*(\boldsymbol{a})\colon \mathscr{D}\mathrm{er}_{\boldsymbol{a}} \xrightarrow{\sim} \mathscr{D}\mathrm{er}_{\boldsymbol{\psi}(\boldsymbol{a})}$ of (1.3.2) gives us a bijective map (the isomorphism symbol below, for the moment, denotes an isomorphism of sets)

$$\boldsymbol{\psi}_*\colon T(V) \xrightarrow{\sim} T(W) \tag{2.3.1}$$

given by $D \mapsto \boldsymbol{\psi}_*(\varpi(D))(D)$ for $D \in T(V)$.

We clearly have a commutative diagram

$$\begin{array}{ccc} T(V) & \xrightarrow[\boldsymbol{\psi}_*]{\sim} & T(W) \\ {\scriptstyle \varpi}\downarrow & & \downarrow{\scriptstyle \varpi} \\ V & \xrightarrow[\boldsymbol{\psi}]{\sim} & W \end{array} \tag{2.3.2}$$

We also have isomorphisms (of sets) $T(V) \xrightarrow{\sim} V \times \mathbf{R}^n$ and $T(W) \xrightarrow{\sim} W \times \mathbf{R}^n$ described in (2.2.1). Let $\boldsymbol{\psi}_\times : V \times \mathbf{R}^n \xrightarrow{\sim} W \times \mathbf{R}^n$ be the bijective map induced by $\boldsymbol{\psi}_*$, i.e. $\boldsymbol{\psi}_\times$ is the unique map making the following diagram commute:

$$\begin{array}{ccc} T(V) & \xrightarrow[\boldsymbol{\psi}_*]{\sim} & T(W) \\ {\scriptstyle (2.2.1)}\downarrow\wr & & \wr\downarrow{\scriptstyle (2.2.1)} \\ V \times \mathbf{R}^n & \xrightarrow[\boldsymbol{\psi}_\times]{\sim} & W \times \mathbf{R}^n \end{array} \tag{2.3.3}$$

From Theorem 1.3.4 it is clear that

$$\boldsymbol{\psi}_\times(\boldsymbol{a}, \boldsymbol{\mu}) = (\boldsymbol{\psi}(\boldsymbol{a}), (J\boldsymbol{\psi})(\boldsymbol{a})(\boldsymbol{\mu})) \tag{2.3.4}$$

and hence we get

Proposition 2.3.5. *The map $\boldsymbol{\psi}_\times$ is a $\mathscr{C}^\infty$ map.*

Proof. From (2.3.4) we have $\boldsymbol{\psi}_\times = (\boldsymbol{\psi} \circ \pi, J\boldsymbol{\psi}(\pi)(\pi_2))$, where $\pi_2 \colon V \times \mathbf{R}^n \to \mathbf{R}^n$ is the projection to the second factor. Since $\boldsymbol{\psi}$, $J\boldsymbol{\psi}$, π, and π_2 are all $\mathscr{C}^\infty$, we are done. □

Another consequence of (2.3.4) is the following change of coordinates theorem for a vector field.

Proposition 2.3.6. *Let $\boldsymbol{X}\colon V \to T(V)$ be a vector field and $\boldsymbol{\mu}_{\boldsymbol{X}}\colon V \to \mathbf{R}^n$ and $\boldsymbol{\mu}_{\boldsymbol{\psi}_*\boldsymbol{X}}\colon W \to \mathbf{R}^n$ be the maps given by (2.2.4). Then*

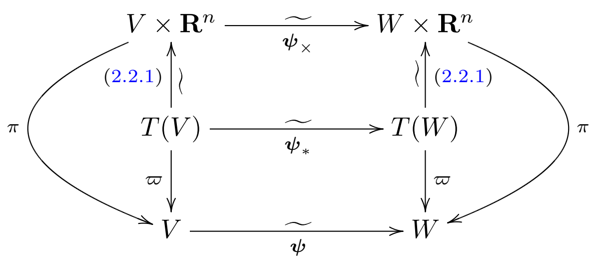
$$\boldsymbol{\mu}_{\boldsymbol{\psi}_*\boldsymbol{X}} = \boldsymbol{\psi}'(\pi)\boldsymbol{\mu}_{\boldsymbol{X}} = J\boldsymbol{\psi}(\pi)\boldsymbol{\mu}_{\boldsymbol{X}}.$$

The following is an obvious corollary of Proposition 2.3.6 and Definition 2.2.5.

Corollary 2.3.7. *If $\boldsymbol{X}$ is a $\mathscr{C}^k$ vector field on V, then $\boldsymbol{\psi}_*\boldsymbol{X}$ is a $\mathscr{C}^k$ vector field on W.*

The situation is summarised in the following commutative diagram:

(2.3.8)



2.3.9. Change of notation. Let k be a non-negative integer. Let $\boldsymbol{X}\colon V \to T(V)$ be a $\mathscr{C}^k$ vector field. We will use the symbol $\boldsymbol{X_a}$ for the tangent vector $\boldsymbol{X}(\boldsymbol{a})$ for $\boldsymbol{a} \in V$. For $f \in \mathscr{C}^\infty(V)$, we write $\boldsymbol{X}(f)$ for the map

$$\begin{aligned} \boldsymbol{X}(f)\colon \mathscr{C}^\infty(V) &\longrightarrow \mathscr{C}^k(V) \\ \boldsymbol{a} &\longmapsto \boldsymbol{X_a}(f). \end{aligned} \tag{2.3.9.1}$$

We leave it to the reader to check that our assertion that $\boldsymbol{X}(f) \in \mathscr{C}^k(V)$ is true. One has to use the fact that $\mathbf{D}_i(f) \in \mathscr{C}^\infty$, $i = 1, \ldots, n$, where $\mathbf{D}_i$ is the smooth vector field $\boldsymbol{a} \mapsto \mathbf{D}_{i,\boldsymbol{a}}$, i.e. $\mathbf{D}_i = \frac{\partial}{\partial x_i}$.

2.3.10. Tweaks. Since $\mathbf{D}_i(f)$ makes sense for $f \in \mathscr{C}^r(V)$ for $r \geq 1$, if $\mathbf{X}$ is a $\mathscr{C}^k$ vector field on V, and $f \in \mathscr{C}^r(V)$ for some $r \geq 1$, one can make sense of $\mathbf{X}(f)$ in this case too. Since $\mathbf{D}_i(f) \in \mathscr{C}^{r-1}(V)$, $i = 1, \ldots, n$, $\mathbf{X}(f)$ will be in $\mathscr{C}^j(V)$ where $j = \min(k, r-1)$. It is easy to see that if this property is true for members of cover consisting of co-ordinate charts, then it is true for any co-ordinate chart.

3. Tangent vectors and vector fields on manifolds

Look up the definition of a differential manifold from any book. (This was Problem 6 of HW 7 of ANA2.) What follows is an obvious generalisation of everything we have discussed. In what follows, a manifold will mean a smooth (i.e. $\mathscr{C}^\infty$) differential manifold. An important definition is the definition of a $\mathscr{C}^k$ map on a differential manifold M. For such an M, a map $f\colon M \to \mathbf{R}$ s said to be $\mathscr{C}^k$ if we can find an atlas $\mathscr{A} = \{(U, \varphi)\}$ of M such that if $(U, \varphi) \in \mathscr{A}$ with $\varphi(U) = V$ (V an open subset of some Euclidean space), then the map $f|_U \circ \varphi^{-1}\colon V \to \mathbf{R}$ is in $\mathscr{C}^k$. It is easy to see that when this happens, if (U, φ) is a co-ordinate chart of M, whether in $\mathscr{A}$ or not, then $f|_U \circ \varphi^{-1}\colon V \to \mathbf{R}$ is in $\mathscr{C}^k$. In the same way, if W is any open set of a Euclidean space, it makes sense to talk of a $\mathscr{C}^k$ map from M to W. Finally, if N is also a differential manifold, then it makes sense to talk about a $\mathscr{C}^k$ map from M to N.[2]

In view of these comments, given $a \in M$, we have the $\mathbf{R}$-algebra $\mathscr{C}^\infty_a$ of germs of $\mathscr{C}^\infty$ functions at a and the space of derivations $\mathscr{D}\mathrm{er}_a = \mathscr{D}\mathrm{er}_a(M)$ of derivations at a. If $D \in \mathscr{D}\mathrm{er}_a$, then D is an $\mathbf{R}$-linear map $D\colon \mathscr{C}^\infty_a \to \mathbf{R}$ such that $D(fg) = f(a)Dg + g(a)Df$. I leave it to you to make sense of $f(a)$ and $g(a)$ for germs f and g at a.

[2] See if you can make these generalisations without looking up a book.

3.1. **Tangent spaces and tangent bundles on a manifold.** . Let M be a manifold of dimension n. For a point $a \in M$, the *tangent space of* M *at* a is the $\mathbf{R}$-vector space $\mathscr{D}\mathrm{er}_a$. It is also denoted T_a, or as $T_a(M)$. As we saw earlier, if M is an open subset of $\mathbf{R}^n$, this agrees with the space of velocity vectors at a, justifying the definition.

The *tangent bundle* of M is the pair $(T(M), \varpi)$ where

$$T(M) = \bigcup_{a \in M} T_a(M) \tag{3.1.1}$$

and $\varpi\colon T(M) \to M$ is the map which sends $D \in \mathscr{D}\mathrm{er}_a = T_a$ to a.

Let $(U_i, \varphi_i\colon U_i \xrightarrow{\sim} G_i)$, $i = 1, 2$ be two co-ordinate charts of M. Note that this means that each G_i is an open subset of $\mathbf{R}^n$(recall, $\dim M = n$). From (2.2.1) we see that we have a set-theoretic bijection $T(U_i) \xrightarrow{\sim} G_i \times \mathbf{R}^n$. Let $U = U_1 \cap U_2$, $V = \varphi_1(U) \subset G_1$, and $W = \varphi_2(U) \subset G_2$. Then we have set-theoretic bijections $T(U) \xrightarrow{\sim} V \times \mathbf{R}^n$ and $T(U) \xrightarrow{\sim} W \times \mathbf{R}^n$, again given by (2.2.1). Moreover, we have a transition function, a diffeomorphism

$$\boldsymbol{\psi}\colon V \xrightarrow{\sim} W.$$

From the commutative diagram (2.3.3), it is easy to see that we have a commutative diagram

$$\begin{array}{ccccc} & & T(U) & & \\ & \swarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\searrow & \\ T(V) & & \xrightarrow[\boldsymbol{\psi}_*]{\sim} & & T(W) \\ {\scriptstyle (2.2.1)}\downarrow\wr & & & & \wr\downarrow{\scriptstyle (2.2.1)} \\ V \times \mathbf{R}^n & & \xrightarrow[\boldsymbol{\psi}_\times]{\sim} & & W \times \mathbf{R}^n \end{array} \tag{3.1.2}$$

From the above considerations, it is clear that $T(M)$ can also be given a manifold structure. To begin with, for each coordinate chart $(\mathfrak{U}, \varphi)$ of M, the set-theoretic bijection $T(\mathfrak{U}) \xrightarrow{\sim} \varphi(\mathfrak{U}) \times \mathbf{R}^n$ gives the structure of a differentiable manifold on $T(\mathfrak{U})$, and the diagram (3.1.2) shows that these differentiable structures are compatible and patch to give a structure of a differential manifold on $T(M)$.

3.1.3. Exercise. Work out and justify the above statements. Show that the map $\varpi\colon T(M) \to M$ is $\mathscr{C}^\infty$.

Definition 3.1.4. A $\mathscr{C}^k$ *vector field on* M is a $\mathscr{C}^k$ section $\boldsymbol{X}\colon M \to T(M)$ of ϖ. In other words, the map $\boldsymbol{X}$ is a $\mathscr{C}^k$ map, and $\varpi \circ \boldsymbol{X} = \mathbf{1}_M$.

The following commutative diagram may help.

$$\begin{array}{ccc} M & \overset{\boldsymbol{X}}{\hookrightarrow} & T(M) \\ & \searrow\!\!\!\searrow & \downarrow{\scriptstyle \varpi} \\ & & M \end{array}$$