MU2202 Ordinary Differential Equations
Semester 2, 2020-21
Mid-Term Exam Solutions

Basic problems. Solve the following. If no initial values are given, give the general solution. Otherwise follow the intsructions.

1) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+3x\dfrac{\mathrm{d}y}{\mathrm{d}x}-35y=x$, $x>0$.

Solution: After using the transformation $t=\ln x$, the DE can be re-written in terms of t as $\ddot{y}+2\dot{y}-35y=e^t$. The complementary solution is $y_c=c_1e^{5t}+c_2e^{-7t}$ where c_1 and c_2 are arbitrary constants. Since e^t is not identically either e^{5t} or e^{-7y}, therefore a particular solution will be of the form $y_p=Ae^t$ for some constant A. A straightforward computation shows that $A=-1/32$. Thus a general solution is $y=c_1x^5+c_2x^{-7}-(1/32)x$, where the c_i's are arbitrary constants. □

2) $\dfrac{\mathrm{d}^4y}{\mathrm{d}t^4}-16y=-30e^t$, $y(0)=3$, $y'(0)=0$, $y''(0)=6$, and $y^{(3)}(0)=-6$.

Solution: The complementary solution is (with the c_i's arbitrary constants)

$$y_c=c_1e^{2t}+c_2e^{-2t}+c_3\cos(2t)+c_4\sin(2t).$$

One can pick the form of a particular solution to be $y_p=Ae^t$. A straightforward computation shows that $A=2$. The general solution is of the form $y=y_c+2e^t$. Working out y,y',y'',y''', and solving the equations $y'(0)=0$, $y''(0)=6$, and $y^{(3)}(0)=-6$, we see that the required solution is $y=e^{-2t}+2e^t$. □

3) $\dot{y}=y(y-1)$, $y(0)=2$. Also give the maximal interval of existence for the IVP.

Solution: The equation is separable and we see that $t=\int y(y-1)^{-1}dy$, i.e. $t=\ln|(y-1)y^{-1}|+C$. Since our initial phase is $y=2$, and the connected component containing 2 of the the open set where vector field $v(y)=y(y-1)\neq 0$ is $(1,\infty)$, we see that $t=\ln((y-1)y^{-1})+C$, i.e. we can dispense with the absolute value sign in the interval of interest to us. Doing the usual manipulations, we find that the solution is $y=\frac{2}{2-e^t}$. Examining the denominator, we see that the maximal interval of existence is $(-\infty,\ln 2)$. As an aside, note that as t approaches $\omega_+=\ln 2$, the solution approaches ∞, exactly as predicted by the theory (see Corollary1.1.3 of Lecture 6). □

One-parameter groups. Let $v\colon \mathbf{R}\to\mathbf{R}$ be the map $v(x)=x^3-x$, $x\in\mathbf{R}$ and set $U:=(-1,1)$, $U^+:=(1,\infty)$, $U^-:=(-\infty,-1)$.

General Computations: For problems **4), 5)** and **6)** it is simpler to just first find the solution of the DE $\dot{x}=x^3-x$ when the initial state is x_0. The same computations help with all three problems. Note that the regular locus of v is $\Omega^{\text{reg}}=\mathbf{R}\setminus\{-1,0,1\}$ and hence if x_0 lies in any one the four connected components of Ω^{reg}, then the solution with x_0 as the initial phase takes values in that interval (you do not have to mention this in your solution, but I am putting it out there to help understanding).The DE is separable. We have $t=\int(x^3-x)dx$. The relevant partial fraction decomposition is

$$\frac{1}{x^3-x}=\frac{1}{2}\frac{1}{x+1}-\frac{1}{x}+\frac{1}{2}\frac{1}{x-1}.$$

This gives

$$t=\ln\left\{\frac{\sqrt{|x^2-1|}}{|x|}\right\}-\ln\left\{\frac{\sqrt{|x_0^2-1|}}{|x_0|}\right\}\qquad(*)$$

For $x_0 \notin \{-1, 0, 1\}$ we conclude that:

$$e^t = \frac{|x_0|}{\sqrt{|x_0^2 - 1|}} \frac{\sqrt{|x^2 - 1|}}{|x|}. \tag{\dagger}$$

4) Find a one-parameter group of diffeomorphisms $\{g^t\}$ on U whose phase velocity field is $v|_U$. ("Find" means (a) give an explicit formula for $g^t x$ for $x \in U$ and $t \in \mathbf{R}$; (b) show that $g^t x \in U$ for every $x \in U$; and (c) argue that $\{g^t\}$ is a one-parameter group of diffeomorphisms.)

Solution: If $x_0 \in (-1, 0) \cup (0, 1)$ then $|x_0^2 - 1| = 1 - x_0^2$ and since $x(t)$ will lie in this set in the interval of existence if $x(0) = x_0$, we also have $|x^2 - 1| = 1 - x^2$. Solving for x from the equation ($\dagger$), we get:

$$x = \frac{x_0}{\sqrt{(1 - x_0^2)e^{2t} + x_0^2}}.$$

We have used the fact that $x_0/x = |x_0/x|$ if $x_0 \in (-1, 0) \cup (0, 1)$. Since $v(0) = 0$, we know that if $x_0 = 0$, then the unique solution to our DE is $x \equiv 0$. So the above formula in fact works for all of U, including when $x_0 = 0$. The denominator is never zero for $x_0 \in U$ since $1 - x_0^2 > 0$ for $x_0 \in U$. It follows that the interval of existence is $\mathbf{R}$ for $x_0 \in U$. Now define, for $t \in \mathbf{R}$, $g\colon \mathbf{R} \times U \to U$ by the formula

$$g(t, x) = \frac{x}{\sqrt{(1 - x^2)e^{2t} + x^2}}.$$

It is clear that g is $\mathscr{C}^1$. We have to check that $g^t(g^s x) = g^{t+s} x$, where the notation is as in the lectures and homework assignments. There are two ways from here, both easy. The first way is to compute; in other words show that

$$\frac{g^s x}{\sqrt{(1 - (g^s x)^2)e^{2t} + (g^s x)^2}} = \frac{x}{\sqrt{(1 - x^2)e^{2(t+s)} + x^2}}$$

where of course $g^s x = \frac{x}{\sqrt{(1-x^2)e^{2s}+x^2}}$. This is surprisingly easy as many of you probably discovered. The second way is to argue as you might for Problem **3)** of HW 6. Since that HW was not submitted, you are expected to give an argument in the exam if you take that route. $\square$

Note: Many of the things done here are not necessary to get full credit for the problem. All you have to do is (a) write out the formula for $g^t x$; (b) verify that your formula gives the solution of the DE when the initial state is x and note that as a function of two variables g is $\mathscr{C}^1$; (c) either by computing or by arguing using uniqueness of solutions, show that $g^t g^s = g^{t+s}$. You do not have to tell us how you got to the formula for $g^t x$.

5) Show that there is no one-parameter family of diffeomorphisms on U^+ whose phase velocity is $v|_{U^+}$. [**Hint:** Suppose $x_0 \in U^+$, and $\theta_\infty := \ln\Big\{x_0(x_0^2-1)^{-1/2}\Big\}$. Examine the solution of the IVP ($\dot{x} = v(x)$, $x(0) = x_0$), as $t \to \theta_\infty$.]

Solution: Note that a necessary condition for a one-parameter group with phase field v to act on U^+ is that the maximal intervals of existence for the IVP's $\dot{x} = v(x)$, $x(0) = x_0$, be $\mathbf{R}$ for $x_0 \in U^+$. Observe that θ_∞ is positive, since $\sqrt{x_0^2-1} < x_0$ when $x_0 \in U^+$. Now consider the relation $(*)$ above. If we let $t \nearrow \theta_\infty$ then clearly

$$\ln\left\{\frac{\sqrt{x^2-1}}{x}\right\} \nearrow 0.$$

This implies $\dfrac{x^2-1}{x^2} \nearrow 1$, i.e. $x \nearrow \infty$. Thus the maximal interval of existence of the IVP in this case is not $\mathbf{R}$, and this is clearly a necessary condition for a one-parameter group to exist.

There is another way to approach the problem. It is straightforward from $(\dagger)$ to see that the solution to the IVP $\dot{x} = v(x)$, $x(0) = x_0$ when $x_0 \notin \{-1, 0, 1\}$ is

$$x = \frac{x_0}{\sqrt{|(1-x_0^2)e^{2t} + x_0^2|}}. \qquad (\spadesuit)$$

The denominator is never zero if $|x_0| < 1$ as is easily checked (if $f(t) = (1-x_0^2)e^{2t} + x_0^2$, then $f'(t) = 2(1-x_0^2)e^{2t}$, whence $f'(t) > 0$ when $|x_0| < 1$, i.e. f is increasing in this case $\ldots$). However, if $|x_0| > 1$, then $(1-x_0^2)e^{2t} + x_0^2 = 0$ has a solution, namely $t = \frac{1}{2}\ln\left\{\frac{x_0^2}{x_0^2-1}\right\}$, i.e. $t = \theta_\infty$. As an aside, it is worth pointing out that $(\spadesuit)$ gives us solutions in all cases, including the case when $x_0 \in \{-1, 0, 1\}$, as is easily checked. $\square$

6) Show that the solution of ($\dot{x} = v(x)$, $x(0) = -x_0$) is the negative of the solution of ($\dot{x} = v(x)$, $x(0) = x_0$). Is there a one-parameter family of diffeomorphisms on U^- whose phase velocity is $v|_{U^-}$?

Solution: We will use the fact that $v(-x) = -v(x)$ for our v. Let $\varphi\colon (\omega_-, \omega_+) \to \mathbf{R}$ be the maximal solution of $\dot{x} = v(x)$, $x(0) = -x_0$. Let $\psi\colon (\omega_-, \omega_+) \to \mathbf{R}$ be given by the formula $\psi(t) = -\varphi(t)$. Then $\dot{\psi}(t) = -\dot{\varphi}(t) = -v(\varphi(t)) = v(-\varphi(t)) = v(\psi(t))$. Thus ψ is a solution of the DE $\dot{x} = v(x)$. Moreover, $\psi(0) = -\varphi(0) = -x_0$. This solves the first part of the problem. For the second part note that $U^- = \{x \in \mathbf{R} \mid -x \in U^+\}$. If a one-parameter group of diffeomorhisms $\{g^t\}$ with phase field $v|_{U^-}$ acts on U^-, then we have a one parameter group of diffeomorphisms $\{h^t\}$ acting on U^+, where $h^t x = -g^t(-x)$, $x \in U^+, t \in \mathbf{R}$. However we know from the previous problem that this is not possible. $\square$

Maximal interval of existence. In what follows we regard the space $M_{m,n}(\mathbf{R})$, the space of $m \times n$ real matrices, as a Euclidean space in the usual manner. As usual, if $m = n$ we write $M_n(\mathbf{R})$ instead of $M_{n,n}(\mathbf{R})$. The set of invertible $n \times n$ matrices is, as usual, denoted $GL(n, \mathbf{R})$, and is an open subset of the Euclidean space $M_n(\mathbf{R})$, for it is the locus of points on which the continuous function $\det\colon M_n(\mathbf{R}) \to \mathbf{R}$ does not vanish, where for an $n \times n$ matrix A, $\det(A)$ denotes the determinant of A.

7) Let U be an open subset of $\mathbf{R}^n$ and $\boldsymbol{v}\colon U \to \mathbf{R}^n$ a $\mathscr{C}^1$ map. Suppose there exists $\varepsilon > 0$ such that for every $\boldsymbol{a} \in U$ the IVP

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x}(0) = \boldsymbol{a} \tag{$(*)_{\boldsymbol{a}}$}$$

has a solution on $(-\varepsilon, \varepsilon)$. Show that the maximal interval of existence for $(*)_{\boldsymbol{a}}$ is $\mathbf{R}$ for every $\boldsymbol{a} \in U$.

Solution: Let $\boldsymbol{a} \in U$, and let the maximal interval of existence for $(*)_{\boldsymbol{a}}$ be $(\omega_-, \omega_+) = (\omega_-(\boldsymbol{a}), \omega_+(\boldsymbol{a}))$. Suppose $\omega_+ < \infty$. Let $\tau = \omega_+ - \frac{1}{2}\varepsilon$ and $\boldsymbol{b} = \boldsymbol{\varphi}_{\boldsymbol{a}}(\tau)$. (Since the length of (ω_-, ω_+) is at least 2ε, τ is in the interval of existence for $(*)_{\boldsymbol{a}}$ and so $\boldsymbol{b}$ makes sense.) Let $\boldsymbol{\psi}$ be the U-valued map defined in a neighbourhood of τ by the formula $\boldsymbol{\psi}(t) = \boldsymbol{\varphi}_{\boldsymbol{b}}(t - \tau)$. Then $\boldsymbol{\psi}$ exists (at least) on $(\tau - \varepsilon, \tau + \varepsilon)$ and is a solution to the IVP $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}(\tau) = \boldsymbol{b}$. On the other hand so is $\boldsymbol{\varphi}_{\boldsymbol{a}}$. Since $\boldsymbol{v}$ is $\mathscr{C}^1$ it is locally Lipschitz and hence by local uniqueness, $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}_{\boldsymbol{a}}$ agree on the intersection of their domains. This means $\boldsymbol{\varphi}_{\boldsymbol{a}}$ can be extended to $(\omega_-, \tau + \varepsilon) = (\omega_-, \omega_+ + \frac{1}{2}\varepsilon)$. This is a contradiction, whence $\omega_+ = \infty$. Similarly $\omega_- = -\infty$. □

8) Fix $A \in M_n(\mathbf{R})$, and let $\boldsymbol{v}\colon GL(n, \mathbf{R}) \to M_n(\mathbf{R})$ be the map given by $\boldsymbol{v}(S) = AS$, $S \in GL(n, \mathbf{R})$. For $S_0 \in GL(n, \mathbf{R})$, let $(*)_{S_0}$ denote the IVP

$$\dot{S} = \boldsymbol{v}(S), \qquad S(0) = S_0. \tag{$(*)_{S_0}$}$$

Show, *without using exponentials*, that for every $S_0 \in GL(n, \mathbf{R})$, the maximal interval of existence for $(*)_{S_0}$ is $\mathbf{R}$. [**Hint:** Relate solutions of $(*)_{S_0}$ with solutions of $(*)_{I_n}$, where I_n is the identity $n \times n$ matrix. Use Problem 7).]

Solution: Let ε be a positive number such that $(-\varepsilon, \varepsilon)$ is an interval of existence for $(*)_{I_n}$. Let J be an interval containing 0 and $\varphi\colon J \to GL(n, \mathbf{R})$ a map. It is easy to see that φ is a solution of $(*)_{I_n}$ if and only if $\psi\colon J \to GL(n, \mathbf{R})$ given by $\psi(t) = \varphi(t)S_0$, is a solution of $(*)_{S_0}$. Indeed $\dot{\psi}(t) = \dot{\varphi}(t)S_0$ for $t \in J$, and hence to say that $\dot{\psi}(t) = A(t)\psi(t)$, $t \in J$, is the same as saying $\dot{\varphi}(t)S_0 = A(t)\varphi(t)S_0$, $t \in J$, and since S_0 is invertible, this is equivalent to saying $\dot{\varphi}(t) = A(t)\varphi(t)$, $t \in J$. Moreover, $\psi(0) = S_0$ if and only if $\varphi(0) = I_n$. Thus every interval of existence of $(*)_{I_n}$ is an interval of existence of $(*)_{S_0}$. It follows that $(-\varepsilon, \varepsilon)$ is an interval of existence for $(*)_{S_0}$ for every $S_0 \in GL(n, \mathbf{R})$. By Problem **7)** we are done. □

Linear Differential Equations. Let I be an interval (for definiteness assume I is open, though this condition is not necessary). Let $A\colon I \to M_n(\mathbf{R})$ and $\boldsymbol{g}\colon I \to \mathbf{R}^n$ be continuous functions. For $(\tau, \boldsymbol{x}) \in I \times \mathbf{R}^n$ let $\boldsymbol{\varphi}_{(\tau,\boldsymbol{x})}\colon I \to \mathbf{R}^n$ be the unique solution to

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + \boldsymbol{g}, \qquad \boldsymbol{y}(\tau) = \boldsymbol{x} \qquad (\dagger)_{(\tau,\boldsymbol{x})}.$$

Let $\boldsymbol{F}\colon I \times \mathbf{R}^n \times I \to \mathbf{R}^n$ be the map given by the formula:

$$\boldsymbol{F}(\tau, \boldsymbol{x}, t) = \boldsymbol{\varphi}_{(\tau,\boldsymbol{x})}(t), \qquad (\tau, \boldsymbol{x}, t) \in I \times \mathbf{R}^n \times I.$$

Let $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_n$ be a basis of solutions for the homogeneous differential equation $\dot{\boldsymbol{y}} = A\boldsymbol{y}$ and M the $n \times n$ matrix of functions whose i^{th} column is $\boldsymbol{\psi}_i$ for $i = 1, \dots, n$. The aim of the next two exercises is to show that $\boldsymbol{F}$ is $\mathscr{C}^1$. Note that the partial derivative of $\boldsymbol{F}$ with respect to t exists and by definition is $\dot{\boldsymbol{\varphi}}_{(\tau,\boldsymbol{x})}(t)$ at $(\tau, \boldsymbol{x}, t)$ and hence is continuous, for $\dot{\boldsymbol{\varphi}}_{(\tau,\boldsymbol{x})}(t) = A(t)\boldsymbol{\varphi}_{(\tau,\boldsymbol{x})}(t) + \boldsymbol{g}(t)$.

General Computations: We know that solutions of the homogeneous equation $\dot{\boldsymbol{y}} = A\boldsymbol{y}$ are all of the form $t \mapsto M(t)\boldsymbol{c}$ where $\boldsymbol{c}$ is a constant vector in $\mathbf{R}^n$, since solutions are linear combinations of $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_n$. In fact the representation $t \mapsto M(t)\boldsymbol{c}$ is the unique solution to the IVP $\dot{\boldsymbol{y}} = A\boldsymbol{y}$, $y(\tau) = M(\tau)\boldsymbol{c}$. If we set $\boldsymbol{Y}_c(t) = M(t)M(\tau)^{-1}\boldsymbol{x}$, then $\boldsymbol{Y}_c$ is the solution of the IVP $\dot{\boldsymbol{y}} = A\boldsymbol{y}$, $\boldsymbol{y}(\tau) = \boldsymbol{x}$. By the technique of variation of parameters we know that $\boldsymbol{Y}_p\colon I \to \mathbf{R}^n$ given by $\boldsymbol{Y}_p(t) = M(t)\int_\tau^t M(s)^{-1}\boldsymbol{g}(s)ds$ is a particular solution of $\dot{\boldsymbol{y}} = A\boldsymbol{y} + \boldsymbol{g}$ and is such that $\boldsymbol{Y}_p(\tau) = \boldsymbol{0}$. It is clear that $\boldsymbol{\varphi}_{(\tau,\boldsymbol{x})} = \boldsymbol{Y}_c + \boldsymbol{Y}_p$. In other words, we have the formula

$$\boldsymbol{F}(\tau, \boldsymbol{x}, t) = M(t)M(\tau)^{-1}\boldsymbol{x} + M(t)\int_\tau^t M(s)^{-1}\boldsymbol{g}(s)ds. \qquad (**)$$

9) Show that for fixed τ and t, $\boldsymbol{F}(\tau, \boldsymbol{x}, t)$ is $\mathscr{C}^1$ with respect to $\boldsymbol{x} = (x_1, \dots, x_n)$ by showing that that for $i = 1, \dots, n$,

$$\left.\frac{\partial \boldsymbol{F}}{\partial x_i}\right|_{(\tau,\boldsymbol{x},t)}$$

is the unique solution to the IVP

$$\dot{\boldsymbol{\zeta}} = A\boldsymbol{\zeta}, \quad \boldsymbol{\zeta}(\tau) = \mathbf{e}_i$$

where $\mathbf{e}_i$, $i = 1, \dots, n$ form the standard basis of $\mathbf{R}^n$.

Remark: In particular the partial derivatives of $\boldsymbol{F}$ with respect to the variables $x_1, \dots, x_n$ do not depend upon $\boldsymbol{x} = (x_1, \dots, x_n)$. The DE involving $\boldsymbol{\zeta}$ is a special case of the *equation of variations* associated to a DE, something we will study later in the semester.

Solution: Linear transformations are $\mathscr{C}^\infty$ and since τ and t are constant for this problem, $\boldsymbol{F}$ is (as far as $\boldsymbol{x}$ is concerned) a linear transformation plus a constant vector (see $(**)$). Applying $\dfrac{\partial}{\partial x_i}$ to $(**)$ we get

$$\left.\frac{\partial \boldsymbol{F}}{\partial x_i}\right|_{(\tau,\boldsymbol{x},t)} = M(t)M(\tau)^{-1}\mathbf{e}_i.$$

This is exactly the solution to the IVP $\dot{\boldsymbol{\zeta}} = A\boldsymbol{\zeta}$, $\boldsymbol{\zeta}(\tau) = \mathbf{e}_i$.

10) Show that the partial derivative of $\boldsymbol{F}$ with respect to τ exists on $I\times\mathbf{R}^n\times I$ as a continuous function and is given by the formula

$$\left.\frac{\partial \boldsymbol{F}}{\partial \tau}\right|_{(\tau,\boldsymbol{x},t)} = -M(t)M(\tau)^{-1}\dot{\boldsymbol{\varphi}}_{(\tau,\boldsymbol{x})}(\tau).$$

[**Hint:** Use the fact that $\dot{M} = AM$ and the fact that if $B\colon I \to GL(n,\, R)$ is differentiable, then $\frac{\mathrm{d}}{\mathrm{d}t}(B^{-1}) = -B^{-1}\dot{B}B^{-1}$, a fact which can easily be deduced by differentiating the identity $BB^{-1} = I_n$. You can use these two facts without proof.]

Solution: Since $\boldsymbol{\psi}_i$, $i = 1,\ldots,n$ are $\mathscr{C}^1$, M is also $\mathscr{C}^1$. In particular the map $\tau \mapsto M(\tau)$ is $\mathscr{C}^1$, whence so is the map $\tau \mapsto M(\tau)^{-1}$. Further, by the fundamental theorem of Calculus, for fixed t, the map $\tau \mapsto \int_\tau^t M(s)^{-1}\boldsymbol{g}(s)ds = -\int_t^\tau M(s)^{-1}\boldsymbol{g}(s)ds$, is $\mathscr{C}^1$. The derivative of $\tau \mapsto M(\tau)^{-1}$ with respect to τ is (from the hint given) the map $\tau \mapsto M(\tau)^{-1}\dot{M}(\tau)M(\tau)^{-1}$. Since $\dot{M} = AM$, we get that this derivative is $\tau \mapsto M(\tau)^{-1}A(\tau)$. On the other hand, for fixed t, the derivative of the map $\tau \mapsto -\int_t^\tau M(s)^{-1}\boldsymbol{g}(s)ds$ is $\tau \mapsto -M(\tau)^{-1}\boldsymbol{g}(\tau)$. Examining $(**)$, and in view of the above observations, we see that $\boldsymbol{F}$ is differentiable with respect to τ and

$$\left.\frac{\partial \boldsymbol{F}}{\partial \tau}\right|_{(\tau,\boldsymbol{x},t)} = -M(t)M(\tau)^{-1}A(\tau)\boldsymbol{x} - M(t)M(\tau)^{-1}\boldsymbol{g}(\tau).$$

From here the required result follows. □