MU2202 Ordinary Differential Equations
Semester 2, 2020-21
Mid-Term Exam

Fenruary 22, 2021

All problems are worth 10 marks. The first six problems are computational in nature. The remaining four are theoretical (though you might need to compute some general identities). You are expected to do eight problems. The last four problems, i.e. problems 7–10, are **compulsory**. You can attempt at most two problems from the basic problems, i.e. problems 1–3. **Please upload a front sheet with your name and roll number on the top right corner, and a list of problems you have attempted. Leave enough room for the markers to make a table in which they will enter marks.**

Basic problems. Solve the following. If no initial values are given, give the general solution. Otherwise follow the intsructions.

1) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 35y = x$, $x > 0$.

2) $\dfrac{\mathrm{d}^4y}{\mathrm{d}t^4} - 16y = -30e^t$, $y(0) = 3$, $y'(0) = 0$, $y''(0) = 6$, and $y^{(3)}(0) = -6$.

3) $\dot{y} = y(y-1)$, $y(0) = 2$. Also give the maximal interval of existence for the IVP.

One-parameter groups. Let $v\colon \mathbf{R} \to \mathbf{R}$ be the map $v(x) = x^3 - x$, $x \in \mathbf{R}$ and set $U := (-1, 1)$, $U^+ := (1, \infty)$, $U^- := (-\infty, -1)$.

4) Find a one-parameter group of diffeomorphisms $\{g^t\}$ on U whose phase velocity field is $v|_U$. ("Find" means (a) give an explicit formula for $g^t x$ for $x \in U$ and $t \in \mathbf{R}$; (b) show that $g^t x \in U$ for every $x \in U$; and (c) argue that $\{g^t\}$ is a one-parameter group of diffeomorphisms.)

5) Show that there is no one-parameter family of diffeomorphisms on U^+ whose phase velocity is $v|_{U^+}$. [**Hint:** Suppose $x_0 \in U^+$, and $\theta_\infty := \ln\left\{x_0(x_0^2 - 1)^{-1/2}\right\}$. Examine the solution of the IVP ($\dot{x} = v(x)$, $x(0) = x_0$), as $t \to \theta_\infty$.]

6) Show that the solution of ($\dot{x} = v(x)$, $x(0) = -x_0$) is the negative of the solution of ($\dot{x} = v(x)$, $x(0) = x_0$). Is there a one-parameter family of diffeomorphisms on U^- whose phase velocity is $v|_{U^-}$?

Maximal interval of existence. In what follows we regard the space $M_{m,n}(\mathbf{R})$, the space of $m \times n$ real matrices, as a Euclidean space in the usual manner. As usual, if $m = n$ we write $M_n(\mathbf{R})$ instead of $M_{n,n}(\mathbf{R})$. The set of invertible $n \times n$ matrices is, as usual, denoted $GL(n, \mathbf{R})$, and is an open subset of the Euclidean space $M_n(\mathbf{R})$, for it is the locus of points on which the continuous function $\det\colon M_n(\mathbf{R}) \to \mathbf{R}$ does not vanish, where for an $n \times n$ matrix A, $\det(A)$ denotes the determinant of A.

7) Let U be an open subset of $\mathbf{R}^n$ and $\boldsymbol{v}\colon U \to \mathbf{R}^n$ a $\mathscr{C}^1$ map. Suppose there exists $\varepsilon > 0$ such that for every $\boldsymbol{a} \in U$ the IVP

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x}(0) = \boldsymbol{a} \tag{$(*)_{\boldsymbol{a}}$}$$

has a solution on $(-\varepsilon, \varepsilon)$. Show that the maximal interval of existence for $(*)_{\boldsymbol{a}}$ is $\mathbf{R}$ for every $\boldsymbol{a} \in U$.

8) Fix $A \in M_n(\mathbf{R})$, and let $\boldsymbol{v}\colon GL(n, \mathbf{R}) \to M_n(\mathbf{R})$ be the map given by $\boldsymbol{v}(S) = AS$, $S \in GL(n, \mathbf{R})$. For $S_0 \in GL(n, \mathbf{R})$, let $(*)_{S_0}$ denote the IVP

$$\dot{S} = \boldsymbol{v}(S), \qquad S(0) = S_0. \tag*{$(*)_{S_0}$}$$

Show, *without using exponentials*, that for every $S_0 \in GL(n, \mathbf{R})$, the maximal interval of existence for $(*)_{S_0}$ is $\mathbf{R}$. [**Hint:** Relate solutions of $(*)_{S_0}$ with solutions of $(*)_{I_n}$, where I_n is the identity $n \times n$ matrix. Use Problem **7**).]

Linear Differential Equations. Let I be an interval (for definiteness assume I is open, though this condition is not necessary). Let $A\colon I \to M_n(\mathbf{R})$ and $\boldsymbol{g}\colon I \to \mathbf{R}^n$ be continuous functions. For $(\tau, \boldsymbol{x}) \in I \times \mathbf{R}^n$ let $\boldsymbol{\varphi}_{(\tau,\boldsymbol{x})}\colon I \to \mathbf{R}^n$ be the unique solution to

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + \boldsymbol{g}, \qquad \boldsymbol{y}(\tau) = \boldsymbol{x} \tag*{$(\dagger)_{(\tau,\boldsymbol{x})}$.}$$

Let $\boldsymbol{F}\colon I \times \mathbf{R}^n \times I \to \mathbf{R}^n$ be the map given by the formula:

$$\boldsymbol{F}(\tau, \boldsymbol{x}, t) = \boldsymbol{\varphi}_{(\tau,\boldsymbol{x})}(t), \qquad (\tau, \boldsymbol{x}, t) \in I \times \mathbf{R}^n \times I.$$

Let $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n$ be a basis of solutions for the homogeneous differential equation $\dot{\boldsymbol{y}} = A\boldsymbol{y}$ and M the $n \times n$ matrix of functions whose i^{th} column is $\boldsymbol{\psi}_i$ for $i = 1, \ldots, n$. The aim of the next two exercises is to show that $\boldsymbol{F}$ is $\mathscr{C}^1$. Note that the partial derivative of $\boldsymbol{F}$ with respect to t exists and by definition is $\dot{\boldsymbol{\varphi}}_{(\tau,\boldsymbol{x})}(t)$ at $(\tau, \boldsymbol{x}, t)$ and hence is continuous, for $\dot{\boldsymbol{\varphi}}_{(\tau,\boldsymbol{x})}(t) = A(t)\boldsymbol{\varphi}_{(\tau,\boldsymbol{x})}(t) + \boldsymbol{g}(t)$.

9) Show that for fixed τ and t, $\boldsymbol{F}(\tau, \boldsymbol{x}, t)$ is $\mathscr{C}^1$ with respect to $\boldsymbol{x} = (x_1, \ldots, x_n)$ by showing that that for $i = 1, \ldots, n$,

$$\left.\frac{\partial \boldsymbol{F}}{\partial x_i}\right|_{(\tau,\boldsymbol{x},t)}$$

is the unique solution to the IVP

$$\dot{\boldsymbol{\zeta}} = A\boldsymbol{\zeta}, \quad \boldsymbol{\zeta}(\tau) = \mathbf{e}_i$$

where $\mathbf{e}_i$, $i = 1, \ldots, n$ form the standard basis of $\mathbf{R}^n$.

Remark: In particular the partial derivatives of $\boldsymbol{F}$ with respect to the variables $x_1, \ldots, x_n$ do not depend upon $\boldsymbol{x} = (x_1, \ldots, x_n)$. The DE involving $\boldsymbol{\zeta}$ is a special case of the *equation of variations* associated to a DE, something we will study later in the semester.

10) Show that the partial derivative of $\boldsymbol{F}$ with respect to τ exists on $I \times \mathbf{R}^n \times I$ as a continuous function and is given by the formula

$$\left.\frac{\partial \boldsymbol{F}}{\partial \tau}\right|_{(\tau,\boldsymbol{x},t)} = -M(t)M(\tau)^{-1}\dot{\boldsymbol{\varphi}}_{(\tau,\boldsymbol{x})}(\tau).$$

[**Hint:** Use the fact that $\dot{M} = AM$ and the fact that if $B\colon I \to GL(n, R)$ is differentiable, then $\dfrac{\mathrm{d}}{\mathrm{d}t}(B^{-1}) = -B^{-1}\dot{B}B^{-1}$, a fact which can easily be deduced by differentiating the identity $BB^{-1} = I_n$. You can use these two facts without proof.]