MU2202 Ordinary Differential Equations
Semester 2, 2020-21
Final Exam

April 26, 2021

Do any five problems out of the six below. Each problem is worth 10 marks. The exam is open book. **Please upload a front sheet with your name and roll number on the top right corner. Leave enough room for the markers to make a table in which they will enter marks.**

Differential equations with R as the phase space.

1) Let $\Omega = (0,\,\infty) \times \mathbf{R} \subset \mathbf{R}^2$ and for $(\tau, a) \in \Omega$, let $(\Delta)_{(\tau,\,a)}$ be the IVP

$$(\Delta)_{(\tau,\,a)} \qquad \dot{x} = \frac{1+x^2}{t}, \quad x(\tau) = a.$$

As usual let $\varphi_{(\tau,\,a)}$ be the solution of $(\Delta)_{(\tau,\,a)}$ and $J(\tau,\,a)$ the maximal interval of existence of $\varphi_{(\tau,\,a)}$.

(a) Show that for $(\tau,\,a) \in \Omega$,

$$J(\tau,\,a) = \Big(\tau \exp\big\{-\tfrac{\pi}{2} - \arctan(a)\big\}, \tau\exp\big\{\tfrac{\pi}{2} - \arctan(a)\big\}\Big).$$

Useful fact: The function arctan takes values in $(-\frac{\pi}{2},\,\frac{\pi}{2})$.

(b) Let, as usual, $\widetilde{\Omega}$ be the open set in $\mathbf{R}^3$ consisting of points $(t,\,\tau,\,a)$ such that $(\tau,\,a) \in \Omega$ and $t \in J(\tau,\,a)$. For $(t,\,\tau,\,a) \in \widetilde{\Omega}$, find a formula for $\varphi_{(\tau,\,a)}(t)$. Verify directly from your formula that $(t,\,\tau,\,a) \mapsto \varphi_{(\tau,\,a)}(t)$ is $\mathscr{C}^1$ on $\widetilde{\Omega}$. You may use the fact that certain common functions like $t \mapsto \ln t$, $t \mapsto \tan t$ etc., are $\mathscr{C}^\infty$ in their natural domains of definition. **Do not** quote results from the course material to show that the map is $\mathscr{C}^1$.

(c) Fix $\tau = \tau_0 \in (0,\,\infty)$. For $(t,\,\tau_0,\,x) \in \widetilde{\Omega}$ let

$$\beta(t,\,x) = \left[\frac{2\tan\{\ln(t/\tau_0) + \arctan(x)\}}{t}\right].$$

For $(\tau_0,\,x) \in \Omega$ consider the linear homogenous initial value problem **(EV)** on the interval $J(\tau_0,\,x)$:

$$\textbf{(EV)} \qquad \dot{z} = \beta(t,\,x)z, \quad z(\tau_0) = 1.$$

To clarify, the unknown function to be found is z. Find the solution $\psi\colon J(\tau_0,\,x) \to \mathbf{R}$ of **(EV)** using a your formula for $\varphi_{(\tau,\,x)}(t)$ in part (b). Justify your answer *without substituting your proposed solution into the equation* **(EV)**. You may quote results from the course material.

Solution: It is clear $\frac{1}{1+x^2}\dot{x} = \frac{1}{t}$, whence $\int_\tau^t \frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}s}\mathrm{d}s = \int_\tau^t \frac{1}{s}\mathrm{d}s$, i.e. $\int_a^x \frac{\mathrm{d}y}{1+y^2} = \int_\tau^t \frac{1}{s}\mathrm{d}s$. Hence $\arctan(x(t)) - \arctan a = \ln t - \ln\tau$. This gives us (writing $\varphi_{(\tau,\,a)}(t)$ for $x(t)$),

$$\arctan\varphi_{(\tau,\,a)}(t) = \ln\frac{t}{\tau} + \arctan a.$$

In other words

$$\varphi_{(\tau,\,a)}(t) = \tan\Big\{\ln\big(\tfrac{t}{\tau}\big) + \arctan(a)\Big\}.$$

Part (a): We have $\arctan\varphi_{(\tau,\,a)}(t) = \ln\frac{t}{\tau} + \arctan a$, which means

$$-\frac{\pi}{2} < \ln\frac{t}{\tau} + \arctan a < \frac{\pi}{2}.$$

Part (a) is immediate.

Part (b): Since $\varphi_{(\tau,\,a)}(t) = \tan\Big\{\ln\big(\frac{t}{\tau}\big) + \arctan(a)\Big\}$, it is clear that the map $F\colon \widetilde{\Omega} \to \mathbf{R}$ given by

$$F(t,\,\tau,\,x) = \varphi_x(t) = \tan\Big\{\ln\big(\tfrac{t}{\tau}\big) + \arctan(x)\Big\}$$

is $\mathscr{C}^1$ on $\widetilde{\Omega}$. Indeed t/τ is $\mathscr{C}^1$ on $(0,\infty)\times(0,\infty)$, $\arctan a$ is $\mathscr{C}^1$ on $\mathbf{R}$, and tan is $\mathscr{C}^1$ on $(-\frac{\pi}{2},\,\frac{\pi}{2})$. This gives part (b).

Part (c): Let $v(t,\,x) = \frac{1+x^2}{t}$ for $(t,\,x)\in\Omega$. Clearly v is $\mathscr{C}^1$. In fact it is $\mathscr{C}^\infty$. $Dv(t,\,x) = (-\frac{1+x^2}{t^2},\,\frac{2x}{t})$. The second component is what we denoted in our lectures by $D_2(t,x)$. (See §§ 3.1, especially (3.1.1) of Lectures 21-22.) Thus

$$D_2(t,x) = \frac{2x}{t}.$$

If we write $A(t,x) = D_2(t,\,\varphi_{(\tau_0,x)}(t))$ then it is obvious that

$$A(t,x) = \beta(t,x).$$

We know from the Remark below Theorem 3.2.2 of Lectures 21-22, and also from the proof of *loc.cit.*, that

$$\zeta(t) = \zeta(t,x) := \frac{\partial F}{\partial x}\Big|_{(t,\tau_0,x)}$$

is the solution of the equation of variation **(EV)**. Now

$$\zeta(t) = \frac{\partial F}{\partial x} = \frac{\sec^2\big\{\ln\big(\frac{t}{\tau_0}\big) + \arctan x\big\}}{1+x^2}.$$

The right side of the above gives us the required solution of **(EV)**. □

Vectors and vector fields on $\mathbf{R}^3$.

2) Let U be an open set in $\mathbf{R}^3$ homeomorphic to an open ball in $\mathbf{R}^3$. Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{v}$ be smooth nowhere vanishing vector fields on U such that $\boldsymbol{u}_1\times\boldsymbol{u}_2$ is also nowhere vanishing on U and such that the inner products $\langle\boldsymbol{u}_1,\,\boldsymbol{v}\rangle$ and $\langle\boldsymbol{u}_2,\,\boldsymbol{v}\rangle$ vanish identically on U. Suppose further that $\mathbf{curl}\,\boldsymbol{u}_1 = \mathbf{curl}\,\boldsymbol{u}_2 = \mathbf{0}$ on U. Given this data, and given $\boldsymbol{a}\in U$, what would be your strategy for solving the initial value problem $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{a}$ in a neighbourhood of the time point 0 (not necessarily on the entire interval of existence)? Write out your procedure carefully, and prove that your procedure ends with a solution of the given IVP. You should make use of the properties of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. You may give references to course notes, but you should make an effort to write out the statement you are using (along with the reference).

Solution: Let $f\colon U\to\mathbf{R}$ and $g\colon U\to\mathbf{R}$ be the potentials of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Since $\boldsymbol{u}_1\perp\boldsymbol{v}$, we see that $\langle\boldsymbol{\nabla}f,\boldsymbol{v}\rangle = 0$. Similarly $\langle\boldsymbol{\nabla}g,\boldsymbol{v}\rangle = 0$. Thus by definition of first integrals, f and g are first integrals for $\boldsymbol{v}$ on U (see Section 1.1 of Lecture 14). Let $\boldsymbol{F} = (f,\,g)$. Then $\boldsymbol{F}$ is a smooth map from U to $\mathbf{R}^2$. It is clear that $J\boldsymbol{F} = \begin{bmatrix}\boldsymbol{u}_1^t\\ \boldsymbol{u}_2^t\end{bmatrix}$, where $(-)^t$ denotes the transpose operation. Now clearly the components of $\boldsymbol{u}_1\times\boldsymbol{u}_2$ are $\pm$ the 2×2 minors of $J\boldsymbol{F}$, and since $\boldsymbol{u}_1\times\boldsymbol{u}_2$ is nowhere vanishing on U, $J\boldsymbol{F}$ has rank 2 everywhere on U. It follows from the implicit function theorem that $\boldsymbol{F}^{-1}(\boldsymbol{c})$ is a one-dimensional smooth manifold for every $\boldsymbol{c}\in\boldsymbol{F}(U)$. Let $\boldsymbol{c} = \boldsymbol{F}(\boldsymbol{a})$ and C the connected component of $\boldsymbol{F}^{-1}(\boldsymbol{c})$ containing $\boldsymbol{a}$. For each $\boldsymbol{p}\in C$ there is neighbourhood U and a diffeomorphism $\psi\colon U \xrightarrow{\sim} I$ into an open interval I, and the inverse of ψ gives a parameterisation of C

in U. One can re-parameterise as in the proof of Theorem 1.2.1 of Lecture 16 to get a solution of $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{p}$. Setting $\boldsymbol{p} = \boldsymbol{a}$ we get a solution as required. The maximal one can be obtained by patching as shown in *loc.cit.* $\square$

3) Let S be the sphere in $\mathbf{R}^3$ whose equation is $x^2 + y^2 + z^2 = 16$. Consider the family of curves $\{C_\nu\}_{\nu>0}$ on S given by the intersection of S with the surface $xy = \nu z$ as ν varies in $(0, \infty)$. Let $\boldsymbol{p} = (\alpha, \beta, \gamma)$ be a point on S in the first octant (i.e. all the co-ordinates of $\boldsymbol{p}$ are positive). Write down an IVP (with $\boldsymbol{v}$ nowhere vanishing on $S \cap$ (first octant))

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{p}$$

such that the solution $\boldsymbol{\varphi}$ takes values in S and such that its trajectory intersects each C_ν it encounters in an orthogonal way (see FIGURE 1 and FIGURE 2 below). Justify your answer. You do not have to solve the IVP. **Hint:** Use cross-products. You may have to use them more than once.

Solution: Taking the gradient of the function $x^2 + y^2 + z^2$, we see that $2(x, y, z)$ is orthogonal to S at each point (x, y, z) of S. Thus the vector field $\boldsymbol{u} = (x, y, z)$ is orthogonal to S at the points of S. In the first octant let f be the function given by $g(x, y, z) = \frac{xy}{z}$. The level surface $S_\nu := g^{-1}(\nu)$ intersects S in the curve C_ν. We wish to make sure that "orthogonal to C_ν" makes sense in the first octant, and for that we need to check that C_ν is a 1-manifold in the first octant. It is enough to check that $\boldsymbol{u}\times\boldsymbol{\nabla} g$ is non-zero on every point of C_ν in the first octant. Note that if this is so, this cross product being orthogonal to both $\boldsymbol{\nabla} g$ and to the gradient of $x^2 + y^2 + z^2$ must be tangential to both S_ν and S, whence is a tangent vector of C_ν. We can work with $\boldsymbol{u}^* = z^2\boldsymbol{\nabla} g$ rather than $\boldsymbol{\nabla} g$. A simple calculation shows that $\boldsymbol{u}^* = z^2\boldsymbol{\nabla} g(\boldsymbol{p}) = (yz, xz, -xy)$. Let $\boldsymbol{w} = \boldsymbol{u}\times\boldsymbol{u}^*$. An easy computation gives

$$\boldsymbol{w} = \begin{bmatrix} -x(y^2+z^2) \\ y(z^2+x^2) \\ z(x^2-y^2) \end{bmatrix}.$$

Note that $\boldsymbol{w} \neq \mathbf{0}$ in the first octant, and hence C_ν is smooth in the first octant. We want a vector field $\boldsymbol{v}$ which when restricted to C_ν, is orthogonal to it, and not tangential to it. Therefore $\boldsymbol{w}$ is not the vector field we want for our IVP. Moreover we would like this vector field $\boldsymbol{v}$ to be tangential to S when restricted to S. Vectors tangential to S at $\boldsymbol{p} \in S$ are the ones orthogonal to $\boldsymbol{u}(\boldsymbol{p})$. It is clear that $\boldsymbol{u}\times\boldsymbol{w}$ satisfies both our requirements. So set $\boldsymbol{v} = \boldsymbol{u}\times\boldsymbol{w}$. One checks easily that

$$\boldsymbol{v} = \begin{bmatrix} -yz(y^2+z^2) \\ -zx(z^2+x^2) \\ xy(x^2+y^2+2z^2) \end{bmatrix}.$$

With $\boldsymbol{v}$ as above and $\boldsymbol{p}$ a point on S in the first octant, we see that the phase curve of $\dot{x} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{p}$, is orthogonal to every C_ν it encounters. $\square$

Stability. Let $\boldsymbol{v}\colon U \to \mathbf{R}^n$ be a locally Lipschitz vector field on an open subset U of $\mathbf{R}^n$. For $\boldsymbol{a} \in U$, let $\boldsymbol{\varphi}_{\boldsymbol{a}}$ be the solution of $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{a}$ and $J(\boldsymbol{a})$ the maximal interval of existence of $\boldsymbol{\varphi}_{\boldsymbol{a}}$. Write $J_{\geqslant 0}(\boldsymbol{a})$ for $J(\boldsymbol{a}) \cap [0, \infty)$. Let Γ be the image of an embedding of the unit $(n-1)$-sphere S^{n-1} into U via a continuous map (embedding means that the map $S^{n-1} \to U$ is injective). It is well known that $\mathbf{R}^n \smallsetminus \Gamma$ has exactly two connected components, a bounded component R, and an unbounded component R'. Moreover the closure of R (resp. R') in $\mathbf{R}^n$ is the union of R (resp. R')and Γ. You may assume this theorem, which is a generalisation of the Jordan Curve Theorem. See FIGURE 3 below.

4) With the above notations, suppose $R \subset U$. Let $\boldsymbol{x}_0 \in R$ be an equilibrium point of $\boldsymbol{v}$. Denote the flow on U determined by $\boldsymbol{v}$ by $\{g^t\}$. In other words, $g^t\boldsymbol{x}$ is defined for $t \in J(\boldsymbol{x})$ and $g^t\boldsymbol{x} = \boldsymbol{\varphi}_{\boldsymbol{x}}(t)$ for $t \in J(\boldsymbol{x})$. Let $F\colon U \to \mathbf{R}$ be a continuous function such that $F(\boldsymbol{x}_0) < F(\boldsymbol{x})$ for all $\boldsymbol{x} \in U \smallsetminus \{\boldsymbol{x}_0\}$. Let γ be the infimum of F on Γ.

(a) Suppose $\boldsymbol{x} \in R$ is such that $F(\boldsymbol{x}) < \gamma$ and such that $t \mapsto F(g^t\boldsymbol{x})$ is a non-increasing function on $J_{\geqslant 0}(\boldsymbol{x})$. Show that $g^t\boldsymbol{x} \in R$ for all $t \in J_{\geqslant 0}(\boldsymbol{x})$. Show also that $J_{\geqslant 0}(\boldsymbol{x}) = [0, , \infty)$.

(b) Suppose $t \mapsto F(g^t\boldsymbol{x})$ is a non-increasing function on $J_{\geqslant 0}(\boldsymbol{x})$ for every $\boldsymbol{x} \in R$. Show that there is an open ball B centred at $\boldsymbol{x}_0$, $B \subset R$, such that if $\boldsymbol{x} \in B$, then $J_{\geqslant 0}(\boldsymbol{x}) = [0, \infty)$ and $g^t\boldsymbol{x} \in R$ for all $t \in J_{\geqslant 0}(\boldsymbol{x})$.

Solution: **Part (a):** If there is a point $s \in J_{\geqslant 0}(\boldsymbol{x})$ such $g^s\boldsymbol{x} \notin R$, then for some $\tau \in J_{\geqslant 0}(\boldsymbol{x})$, $g^\tau\boldsymbol{x} \in \Gamma$. This means $F(g^\tau\boldsymbol{x}) \geq \gamma$, contradicting the fact that $F(\boldsymbol{x}) < \gamma$ and $t \mapsto F(g^t\boldsymbol{x})$ is non-increasing in t for $t \in J_{\geqslant 0}(\boldsymbol{x})$. Thus $g^t\boldsymbol{x} \in R$ for all $t \in J_{\geqslant 0}(\boldsymbol{x})$.

Let $\overline{R}$ be the closure of R in $\mathbf{R}^n$. We know from what we were told that $\overline{R} = R \cup \Gamma$. Since R is bounded $\overline{R}$ is compact. It follows that for every $T > 0$, the set $K_T = [0, T] \times \overline{R}$ is compact, and hence $(t, g^t\boldsymbol{x})$ must exit K_T as $t \uparrow$. Since $g^t\boldsymbol{x} \in R$ for all $t \in J_{\geqslant 0}(\boldsymbol{x})$, it follows that $(t, g^t\boldsymbol{x})$ exits K_T from $\{T\} \times R$. Thus $T \in J_{\geqslant 0}(\boldsymbol{x})$. This proves that $J_{\geqslant 0}(\boldsymbol{x}) = [0, \infty)$.

Part (b): Since F is continuous, there exists $\delta > 0$ such that for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, \delta)$, we have

$$|F(\boldsymbol{x}) - F(\boldsymbol{x}_0)| < \tfrac{1}{2}(\gamma - F(\boldsymbol{x}_0).$$

Now $\boldsymbol{x}_0$ is the unique minima of F and hence $|F(\boldsymbol{x}) - F(\boldsymbol{x}_0)| = F(\boldsymbol{x}) - F(\boldsymbol{x}_0)$. The above inequality then means that $F(\boldsymbol{x}) < \frac{1}{2}(\gamma + F(\boldsymbol{x}_0)) < \gamma$ for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, \delta)$. By part (a), we are done. □

5) Let $F\colon U \to \mathbf{R}$ and γ be as in problem 4). Let $R \subset U$. Suppose $t \mapsto F(g^t\boldsymbol{x})$ is a strictly decreasing function on $J_{\geqslant 0}(\boldsymbol{x})$ for every $\boldsymbol{x} \in R \smallsetminus \{\boldsymbol{x}_0\}$. Show that if $\boldsymbol{x} \in R$ is such that $F(\boldsymbol{x}) < \gamma$ then $\lim_{t\to\infty} g^t x = \boldsymbol{x}_0$.

Solution: If $\boldsymbol{x} = \boldsymbol{x}_0$ there is nothing to prove since in that case $g^t\boldsymbol{x} = g^t\boldsymbol{x}_0 = \boldsymbol{x}_0$ for all $t \in \mathbf{R}$. So assume $\boldsymbol{x} \neq \boldsymbol{x}_0$. By the previous problem, $J_{\geqslant 0}(\boldsymbol{x}) = [0, \infty)$. Let β be the infimum of $F(g^t\boldsymbol{x})$ as t varies in $[0, \infty)$. Then, as $F(g^t\boldsymbol{x})$ is a strictly decreasing function of t in $[0, \infty)$ therefore $F(\boldsymbol{x}_0) \leq \beta < \gamma$. By definition of infimum, we have a sequence of time points

$$0 = t_0 < t_1 < \cdots < t_k < \ldots \uparrow \infty$$

such that $F(g^{t_k}\boldsymbol{x}) \downarrow \beta$ as $k \uparrow \infty$. Since $\{g^{t_k}\boldsymbol{x}\}$ is a sequence in the compact set $\overline{R} = R \cup \Gamma$, it has a convergent subsequence, and by replacing $\{g^{t_k}\boldsymbol{x}\}$ by this convergent subsequence, if necessary, we assume that $\{g^{t_k}\boldsymbol{x}\}$ is convergent. Let

$$\boldsymbol{x}^* = \lim_{k\to\infty} g^{t_k}\boldsymbol{x}.$$

By the continuity of F, we have $F(\boldsymbol{x}^*) = \beta < \gamma$, whence $\boldsymbol{x}^* \in R$. Again by part (a) of the previous problem, this means $J_{\geqslant 0}(\boldsymbol{x}^*) = [0, \infty)$, whence $g^t\boldsymbol{x}^*$ makes sense for all $t \in [0, \infty)$ and $g^t\boldsymbol{x}^* \in R$ for all $t \geq 0$.

Consider the sequence $\{g^{t_k+1}\boldsymbol{x}\}$. Since $g^{t_k+1}\boldsymbol{x} = g^1 g^{t_k}\boldsymbol{x}$, and since $g^1\colon U \to U$ is continuous, we have

$$\lim_{k\to\infty} g^{t_k+1}\boldsymbol{x} = g^1\boldsymbol{x}^*.$$

In particular, since $F(g^{t_k+1}\boldsymbol{x}) < F(g^{t_k}\boldsymbol{x})$ we have, on taking limits

$$F(g^1\boldsymbol{x}^*) \le F(\boldsymbol{x}^*).$$

Now $F(g^{t_k+1}\boldsymbol{x}) \downarrow F(g^1\boldsymbol{x}^*)$, whence, given $\eta > 0$ we can find $K \ge 0$ such that

$$F(g^1\boldsymbol{x}^*) < F(g^{t_k+1}\boldsymbol{x}) < F(g^1\boldsymbol{x}^*) + \eta \qquad (k \ge K).$$

There exists $l \ge 0$ such that $t_l > t_K + 1$. We therefore have the sequence of inequalities

$$F(g^1\boldsymbol{x}^*) \le F(\boldsymbol{x}^*) < F(g^l\boldsymbol{x}) < F(g^{t_K+1}\boldsymbol{x}) < F(g^1\boldsymbol{x}^*) + \eta.$$

Thus, for every $\eta > 0$ we have

$$F(g^1\boldsymbol{x}^*) \le F(\boldsymbol{x}^*) < F(g^1\boldsymbol{x}^*) + \eta.$$

This means $F(g^1\boldsymbol{x}^*) = F(\boldsymbol{x}^*)$. One of our hypotheses is that $t \mapsto F(g^t\boldsymbol{x})$ is strictly decreasing on $[0, \infty)$ for $\boldsymbol{x} \in R \setminus \{\boldsymbol{x}_0\}$. Hence $\boldsymbol{x}^* = \boldsymbol{x}_0$.

Let $\varepsilon > 0$ be given. We have to find $T \ge 0$ such that $g^t\boldsymbol{x} \in B(\boldsymbol{x}_0, \varepsilon)$ for all $t \ge T$. We choose ε so small that the closed ball $\overline{B}(\boldsymbol{x}_0, \varepsilon) \subset R$. With the bounding sphere $S(\boldsymbol{x}_0, \varepsilon)$ playing the role of Γ, part (b) of the previous problem tells us that there is an open ball B centred at $\boldsymbol{x}_0$ such that $B \subset B(\boldsymbol{x}_0, \varepsilon)$ and if $\boldsymbol{y} \in B$, then $J_{\geqslant 0}(\boldsymbol{y}) = [0, \infty)$ and $g^t\boldsymbol{y} \in B(\boldsymbol{x}_0, \varepsilon)$ for all $t \ge 0$. Now there exists $K \ge 0$ such that $g^{t_k}\boldsymbol{x} \in B$ for all $k \ge K$. Then from our just made observation, with $\boldsymbol{y} = g^{t_K}\boldsymbol{x}$, we see that $g^{t+t_K}\boldsymbol{x} = g^t(g^{t_K}\boldsymbol{x}) \in B(\boldsymbol{x}_0, \varepsilon)$ for all $t \ge 0$. Setting $T = t_K$, this means

$$g^t\boldsymbol{x} \in B(\boldsymbol{x}_0, \varepsilon) \qquad (t \ge T).$$

Thus $\lim_{t\to\infty} g^t\boldsymbol{x} = \boldsymbol{x}_0$, as required. □

Miscellaneous.

6) Let Ω be an open subset of $\mathbf{R} \times \mathbf{R}^n$ and $\boldsymbol{v}\colon \Omega \to \mathbf{R}^n$ a map which is Lipschitz in the second variable. Let $(\tau, \boldsymbol{a})$ be a point in Ω, $\boldsymbol{\varphi_0}$ the solution of the initial value problem $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$, $\boldsymbol{x}(\tau) = \boldsymbol{a}$, and J the maximal interval of existence of $\boldsymbol{\varphi_0}$. Let $\{\boldsymbol{v}_m\}$ be a sequence of continuous $\mathbf{R}^n$-valued functions on Ω converging uniformly to $\boldsymbol{v}$. Suppose we have a closed interval $I = [a, b]$ contained in J with $\tau \in I$ such that each of the differential equations $\dot{\boldsymbol{x}} = \boldsymbol{v}_m(t, \boldsymbol{x})$ has a solution $\boldsymbol{\varphi}_m$ on I such that the sequence $\{\boldsymbol{\varphi}_m\}$ satisfies $\lim_{m\to\infty} \boldsymbol{\varphi}_m(\tau) = \boldsymbol{\varphi_0}(\tau)$. Show that $\boldsymbol{\varphi}_m$ converges to $\boldsymbol{\varphi_0}$ uniformly on I as $m \to \infty$.

Solution: Let L be the Lipschitz constant for $\boldsymbol{v}$. Let ε be a positive real number. There exists a positive integer M such that for every $(t, \boldsymbol{x}) \in \Omega$ we have $\|\boldsymbol{v}_m(t, \boldsymbol{x}) - \boldsymbol{v}(t, \boldsymbol{x})\| < \varepsilon$ whenever $m \ge M$. The integer M depends only on ε and not on $(t, \boldsymbol{x})$. For $t \in I$ and $m \ge M$ we then have

$$\|\dot{\boldsymbol{\varphi}}_m(t) - \boldsymbol{v}(t, \boldsymbol{\varphi}_m(t))\| = \|\boldsymbol{v}_m(t, \boldsymbol{\varphi}_m(t)) - \boldsymbol{v}(t, \boldsymbol{\varphi}_m(t))\| \le \varepsilon.$$

Thus for $m \ge M$, $\boldsymbol{\varphi}_m$ is an ε-approximate solution of the DE $\dot{x} = \boldsymbol{v}(t, \boldsymbol{x})$. Applying the fundamental estimate, we see that

$$\|\boldsymbol{\varphi}_m(t) - \boldsymbol{\varphi_0}(t)\| \le \|\boldsymbol{\varphi}_m(\tau) - \boldsymbol{a}\|e^{L(b-a)} + \frac{\varepsilon}{L}\Big(e^{L(b-a)} - 1\Big),$$

for $m \geq M$ and for all $t \in I$. Since $\lim_{m\to\infty} \boldsymbol{\varphi}_m(\tau) = \boldsymbol{\varphi}_{\mathbf{0}}(\tau) = \boldsymbol{a}$, there exists $M' \geq 1$ such that $\|\boldsymbol{\varphi}_m(\tau) - \boldsymbol{a}\| < \varepsilon$ for $m \geq M'$. Letting N be the maximum of M and M' we see that

$$\|\boldsymbol{\varphi}_m(t) - \boldsymbol{\varphi}_{\mathbf{0}}(t)\| \leq \varepsilon\Big(e^{L(b-a)} + L^{-1}(e^{(b-a)} - 1)\Big) \qquad (m \geq N;\, t \in I).$$

Since N does not depend upon $t \in I$, this proves the uniform convergence assertion. $\square$

Below are three pictures. The first two relate to problem **3**). The last one relates to Problems **4**) and **5**)

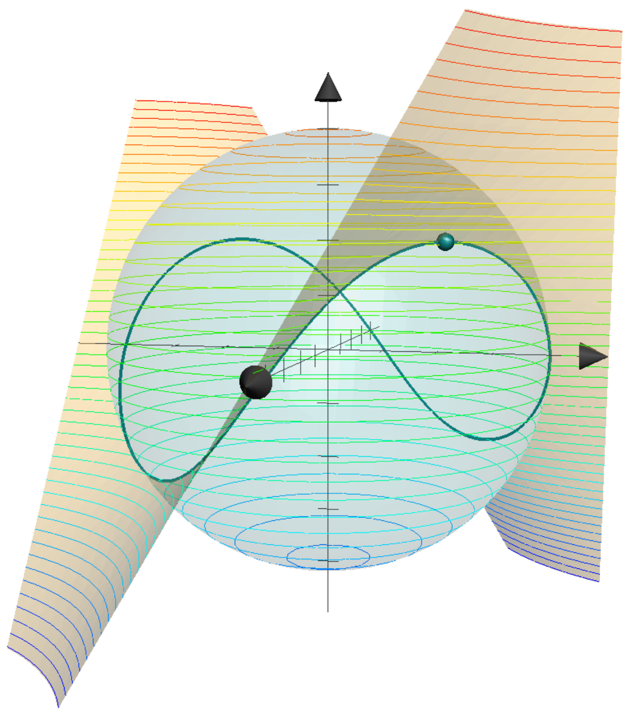

FIGURE 1. The curve C_ν (with $\nu = 3$) realised as the intersection of S with the hyperbolic paraboloid $xy = 3z$.

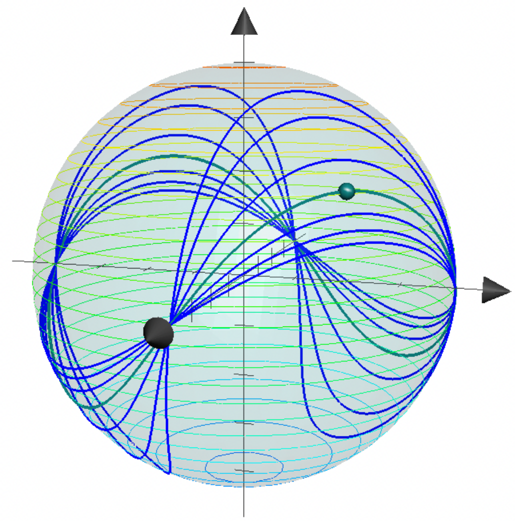

FIGURE 2. Some members of $\{C_\nu\}$. You have been asked to write an autonomous IVP whose initial phase is on the first octant and on the sphere (e.g. the point shown in the picture), such that the trajectory of its solution intersects the members of $\{C_\nu\}$ orthogonally. The family $\{C_\nu\}$ is such that C_ν approaches the circle of radius 4 centred at $\mathbf{0}$ on the xy-plane as $\nu \to \infty$.

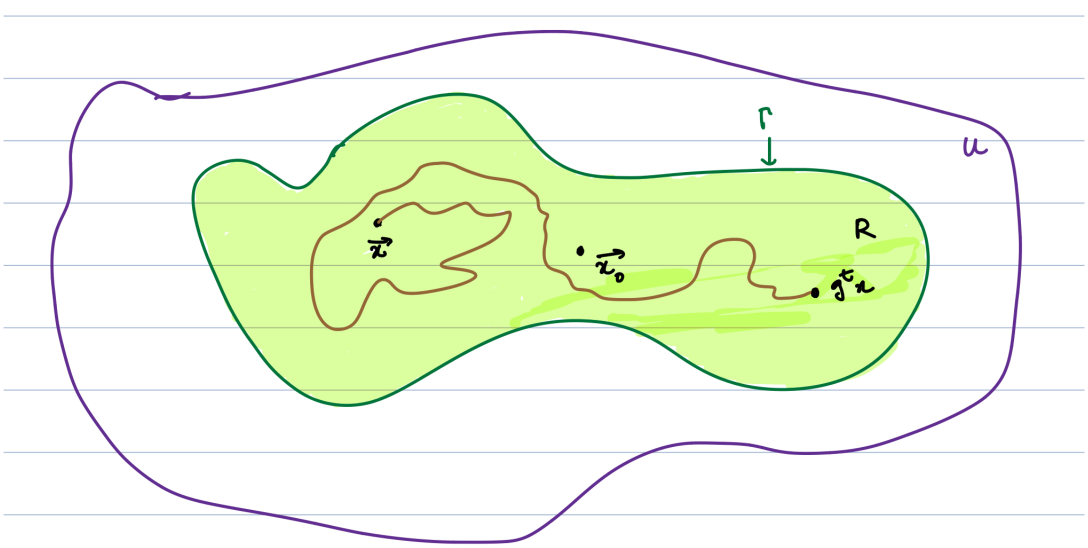

FIGURE 3. Γ is the image of an embedding of S^{n-1} into $\mathbf{R}^n$.