MU2202 Ordinary Differential Equations
Semester 2, 2020-21
Final Exam

April 26, 2021

Do any five problems out of the six below. Each problem is worth 10 marks. The exam is open book. **Please upload a front sheet with your name and roll number on the top right corner. Leave enough room for the markers to make a table in which they will enter marks.**

Differential equations with R as the phase space.

1) Let $\Omega = (0, \infty) \times \mathbf{R} \subset \mathbf{R}^2$ and for $(\tau, a) \in \Omega$, let $(\Delta)_{(\tau, a)}$ be the IVP

$$(\Delta)_{(\tau, a)} \qquad \dot{x} = \frac{1+x^2}{t}, \quad x(\tau) = a.$$

As usual let $\varphi_{(\tau, a)}$ be the solution of $(\Delta)_{(\tau, a)}$ and $J(\tau, a)$ the maximal interval of existence of $\varphi_{(\tau, a)}$.

(a) Show that for $(\tau, a) \in \Omega$,

$$J(\tau, a) = \Big(\tau \exp\big\{-\tfrac{\pi}{2} - \arctan(a)\big\}, \tau \exp\big\{\tfrac{\pi}{2} - \arctan(a)\big\}\Big).$$

Useful fact: The function arctan takes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(b) Let, as usual, $\widetilde{\Omega}$ be the open set in $\mathbf{R}^3$ consisting of points (t, τ, a) such that $(\tau, a) \in \Omega$ and $t \in J(\tau, a)$. For $(t, \tau, a) \in \widetilde{\Omega}$, find a formula for $\varphi_{(\tau, a)}(t)$. Verify directly from your formula that $(t, \tau, a) \mapsto \varphi_{(\tau, a)}(t)$ is $\mathscr{C}^1$ on $\widetilde{\Omega}$. You may use the fact that certain common functions like $t \mapsto \ln t$, $t \mapsto \tan t$ etc., are $\mathscr{C}^\infty$ in their natural domains of definition. **Do not** quote results from the course material to show that the map is $\mathscr{C}^1$.

(c) Fix $\tau = \tau_0 \in (0, \infty)$. For $(t, \tau_0, x) \in \widetilde{\Omega}$ let

$$\beta(t, x) = \left[\frac{2\tan\{\ln(t/\tau_0) + \arctan(x)\}}{t}\right].$$

For $(\tau_0, x) \in \Omega$ consider the linear homogenous initial value problem **(EV)** on the interval $J(\tau_0, x)$:

$$\textbf{(EV)} \qquad \dot{z} = \beta(t, x)z, \quad z(\tau_0) = 1.$$

To clarify, the unknown function to be found is z. Find the solution $\psi\colon J(\tau_0, x) \to \mathbf{R}$ of **(EV)** using a your formula for $\varphi_{(\tau, x)}(t)$ in part (b). Justify your answer *without substituting your proposed solution into the equation* **(EV)**. You may quote results from the course material.

Vectors and vector fields on $\mathbf{R}^3$. Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ be two vectors in $\mathbf{R}^3$. Recall that the *cross-product* $\boldsymbol{u}\times\boldsymbol{v}$ of $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by the formula

$$\boldsymbol{u}\times\boldsymbol{v} = (u_2v_3 - u_3v_2,\, u_3v_1 - u_1v_3,\, u_1v_2 - u_2v_1).$$

Recall further that $\boldsymbol{u}\times\boldsymbol{v}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$ and is zero if and only if the rank of the 3×2 matrix $\begin{bmatrix}\boldsymbol{u} & \boldsymbol{v}\end{bmatrix}$ is strictly less than 2, i.e. if and only if the set $\{\boldsymbol{u}, \boldsymbol{v}\}$ is linearly dependent.

Let $\mathbf{D}_i$, $i = 1, \ldots, n$ be the standard partial derivative operators $\frac{\partial}{\partial x_i}$ on functions on open subsets of $\mathbf{R}^n$. It is well known (via de Rham's theorem) that if U is an open subset of $\mathbf{R}^3$ homeomorphic to an open ball in $\mathbf{R}^3$ (for example $\mathbf{R}^3$ itself) and P, Q, and R are smooth functions on U, then there exists a smooth function $W\colon U \to \mathbf{R}$ such that $\mathbf{D}_1W = P$, $\mathbf{D}_2W = Q$, $\mathbf{D}_3W = R$ if and only if $\mathbf{D}_2R = \mathbf{D}_3Q$, $\mathbf{D}_3P = \mathbf{D}_1R$, and $\mathbf{D}_1Q = \mathbf{D}_2P$ on U.

The function W (if it exists) is called a *potential* of the vector field (P, Q, R). Note that, by definition, W is a potential if and only if $\boldsymbol{\nabla} W = (P, Q, R)$. In other words if

$$\mathbf{curl}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} := \begin{bmatrix} \mathbf{D}_2 R - \mathbf{D}_3 Q \\ \mathbf{D}_3 P - \mathbf{D}_1 R \\ \mathbf{D}_1 Q - \mathbf{D}_2 P \end{bmatrix}$$

then the equation $\boldsymbol{\nabla} W = (P, Q, R)$, with W the unknown, has a solution if and only if $\mathbf{curl}(P, Q, R) = \mathbf{0}$. Two potentials for a vector field differ by a constant, since the partial derivatives of their difference vanish. You may use all the facts given above for solving the problems below.

2) Let U be an open set in $\mathbf{R}^3$ homeomorphic to an open ball in $\mathbf{R}^3$. Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{v}$ be smooth nowhere vanishing vector fields on U such that $\boldsymbol{u}_1 \times \boldsymbol{u}_2$ is also nowhere vanishing on U and such that the inner products $\langle \boldsymbol{u}_1, \boldsymbol{v}\rangle$ and $\langle \boldsymbol{u}_2, \boldsymbol{v}\rangle$ vanish identically on U. Suppose further that $\mathbf{curl}\, \boldsymbol{u}_1 = \mathbf{curl}\, \boldsymbol{u}_2 = \mathbf{0}$ on U. Given this data, and given $\boldsymbol{a} \in U$, what would be your strategy for solving the initial value problem $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{a}$ in a neighbourhood of the time point 0 (not necessarily on the entire interval of existence)? Write out your procedure carefully, and prove that your procedure ends with a solution of the given IVP. You should make use of the properties of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. You may give references to course notes without proof, but you should make an effort to write out the statement you are using (along with the reference).

3) Let S be the sphere in $\mathbf{R}^3$ whose equation is $x^2 + y^2 + z^2 = 16$. Consider the family of curves $\{C_\nu\}_{\nu>0}$ on S given by the intersection of S with the surface $xy = \nu z$ as ν varies in $(0, \infty)$. Let $\boldsymbol{p} = (\alpha, \beta, \gamma)$ be a point on S in the first octant (i.e. all the co-ordinates of $\boldsymbol{p}$ are positive). Write down an IVP (with $\boldsymbol{v}$ nowhere vanishing on $S \cap$ (first octant))

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{p}$$

such that the solution $\boldsymbol{\varphi}$ takes values in S and such that its trajectory intersects each C_ν it encounters in an orthogonal way (see FIGURE 1 and FIGURE 2 below). Justify your answer. You do not have to solve the IVP. **Hint:** Use cross-products. You may have to use them more than once.

Stability. Let $\boldsymbol{v}\colon U \to \mathbf{R}^n$ be a locally Lipschitz vector field on an open subset U of $\mathbf{R}^n$. For $\boldsymbol{a} \in U$, let $\boldsymbol{\varphi}_{\boldsymbol{a}}$ be the solution of $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{a}$ and $J(\boldsymbol{a})$ the maximal interval of existence of $\boldsymbol{\varphi}_{\boldsymbol{a}}$. Write $J_{\geqslant 0}(\boldsymbol{a})$ for $J(\boldsymbol{a}) \cap [0, \infty)$. Let Γ be the image of an embedding of the unit $(n-1)$-sphere S^{n-1} into U via a continuous map (embedding means that the map $S^{n-1} \to U$ is injective). It is well known that $\mathbf{R}^n \smallsetminus \Gamma$ has exactly two connected components, a bounded component R, and an unbounded component R'. Moreover the closure of R (resp. R') in $\mathbf{R}^n$ is the union of R (resp. R')and Γ. You may assume this theorem, which is a generalisation of the Jordan Curve Theorem. See FIGURE 3 below.

4) With the above notations, suppose $R \subset U$. Let $\boldsymbol{x}_0 \in R$ be an equilibrium point of $\boldsymbol{v}$. Denote the flow on U determined by $\boldsymbol{v}$ by $\{g^t\}$. In other words, $g^t\boldsymbol{x}$ is defined for $t \in J(\boldsymbol{x})$ and $g^t\boldsymbol{x} = \boldsymbol{\varphi}_{\boldsymbol{x}}(t)$ for $t \in J(\boldsymbol{x})$. Let $F\colon U \to \mathbf{R}$ be a continuous function such that $F(\boldsymbol{x}_0) < F(\boldsymbol{x})$ for all $\boldsymbol{x} \in U \smallsetminus \{\boldsymbol{x}_0\}$. Let γ be the infimum of F on Γ.

(a) Suppose $\boldsymbol{x} \in R$ is such that $F(\boldsymbol{x}) < \gamma$ and such that $t \mapsto F(g^t\boldsymbol{x})$ is a non-increasing function on $J_{\geqslant 0}(\boldsymbol{x})$. Show that $g^t\boldsymbol{x} \in R$ for all $t \in J_{\geqslant 0}(\boldsymbol{x})$. Show also that $J_{\geqslant 0}(\boldsymbol{x}) = [0, , \infty)$.

(b) Suppose $t \mapsto F(g^t\boldsymbol{x})$ is a non-increasing function on $J_{\geqslant 0}(\boldsymbol{x})$ for every $\boldsymbol{x} \in R$. Show that there is an open ball B centred at $\boldsymbol{x}_0$, $B \subset R$, such that if $\boldsymbol{x} \in B$, then $J_{\geqslant 0}(\boldsymbol{x}) = [0, \infty)$ and $g^t\boldsymbol{x} \in R$ for all $t \in J_{\geqslant 0}(\boldsymbol{x})$.

5) Let $F\colon U \to \mathbf{R}$ and γ be as in problem 4). Let $R \subset U$. Suppose $t \mapsto F(g^t\boldsymbol{x})$ is a strictly decreasing function on $J_{\geqslant 0}(\boldsymbol{x})$ for every $\boldsymbol{x} \in R \smallsetminus \{\boldsymbol{x}_0\}$. Show that if $\boldsymbol{x} \in R$ is such that $F(\boldsymbol{x}) < \gamma$ then $\lim_{t\to\infty} g^t x = \boldsymbol{x}_0$.

Miscellaneous.

6) Let Ω be an open subset of $\mathbf{R} \times \mathbf{R}^n$ and $\boldsymbol{v}\colon \Omega \to \mathbf{R}^n$ a map which is Lipschitz in the second variable. Let $(\tau, \boldsymbol{a})$ be a point in Ω, $\boldsymbol{\varphi_0}$ the solution of the initial value problem $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x}),\ \boldsymbol{x}(\tau) = \boldsymbol{a}$, and J the maximal interval of existence of $\boldsymbol{\varphi_0}$. Let $\{\boldsymbol{v}_m\}$ be a sequence of continuous $\mathbf{R}^n$-valued functions on Ω converging uniformly to $\boldsymbol{v}$. Suppose we have a closed interval $I = [a, b]$ contained in J with $\tau \in I$ such that each of the differential equations $\dot{\boldsymbol{x}} = \boldsymbol{v}_m(t, \boldsymbol{x})$ has a solution $\boldsymbol{\varphi}_m$ on I such that $\{\boldsymbol{\varphi}_m\}$ satisfies $\lim_{m\to\infty} \boldsymbol{\varphi}_m(\tau) = \boldsymbol{\varphi_0}(\tau)$. Show that $\boldsymbol{\varphi}_m$ converges to $\boldsymbol{\varphi_0}$ uniformly on I as $m \to \infty$.

Below are three pictures. The first two relate to problem **3**). The last one relates to Problems 4) and **5**)

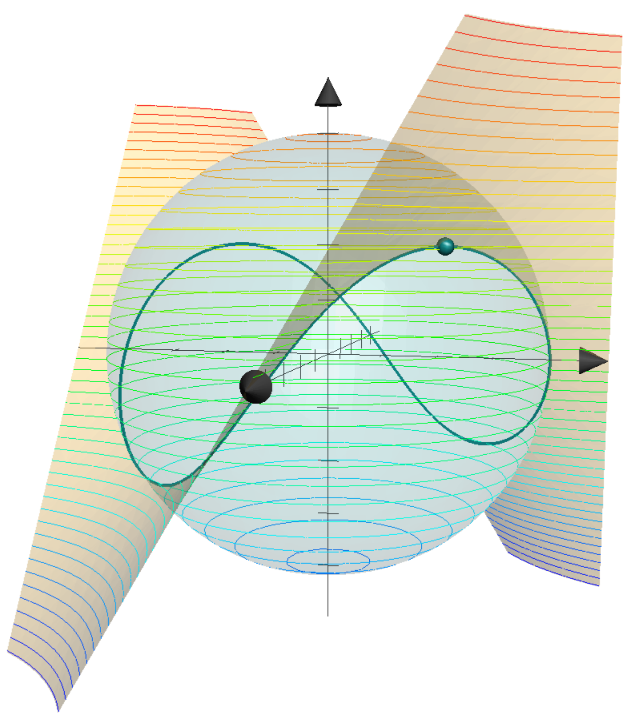

Figure 1. The curve C_ν (with $\nu = 3$) realised as the intersection of S with the hyperbolic paraboloid $xy = 3z$.

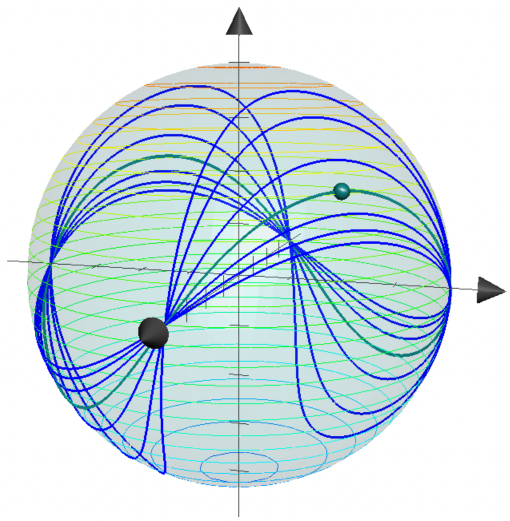

FIGURE 2. Some members of $\{C_\nu\}$. You have been asked to write an autonomous IVP whose initial phase is on the first octant and on the sphere (e.g. the point shown in the picture), such that the trajectory of its solution intersects the members of $\{C_\nu\}$ orthogonally. The family $\{C_\nu\}$ is such that C_ν approaches the circle of radius 4 centred at $\mathbf{0}$ on the xy-plane as $\nu \to \infty$.

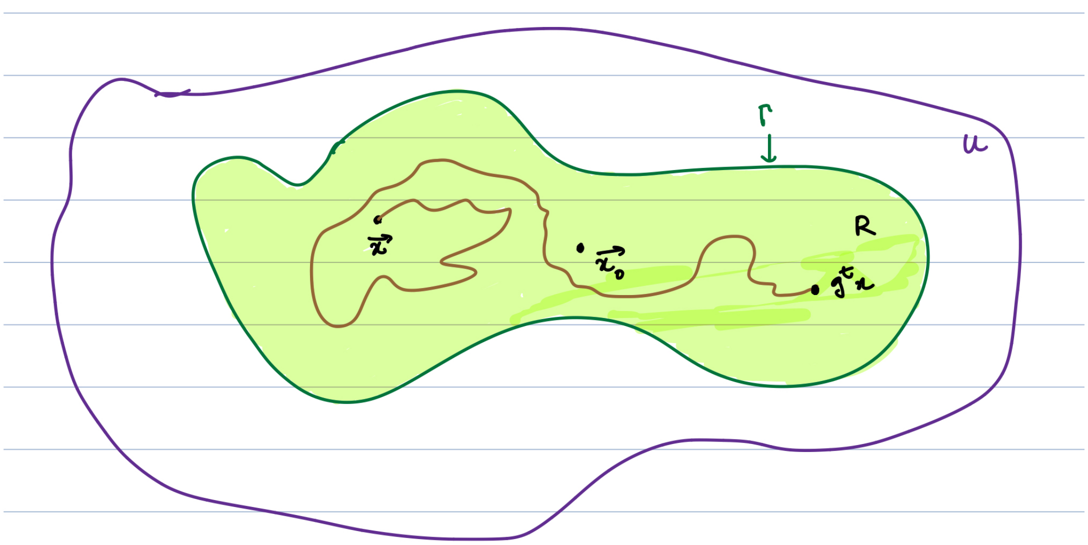

FIGURE 3. Γ is the image of an embedding of S^{n-1} into $\mathbf{R}^n$.