QUIZ 6

Date: April 21, 2021

You may use the following fact (a part of which we have proved in class today).

Theorem: *Let U be an open set in $\mathbf{R}^n$ containing $\mathbf{0}$. Let $\boldsymbol{v}\colon U \to \mathbf{R}^n$ be a $\mathscr{C}^2$ vector field on U and $\boldsymbol{x}_0$ is an equilibrium point of $\boldsymbol{v}$. Let $A = (D\boldsymbol{v})(\boldsymbol{x}_0)$. If A has an eigenvalue whose real part is positive then $\boldsymbol{x}_0$ is an unstable equilibrium point for $\boldsymbol{v}$. If the real parts of all the eigenvalues of A are negative, then $\boldsymbol{x}_0$ is an asymptotically stable equilibrium point of $\boldsymbol{v}$.*

We also give a definition.

Definition: Let $\boldsymbol{x}_0$ be an equilibrium point of the $\mathscr{C}^2$ vector field $\boldsymbol{v}\colon U \to \mathbf{R}^n$, where U is open in $\mathbf{R}^n$. Let $A = (D\boldsymbol{v})(\boldsymbol{x}_0)$. The equilibrium point $\boldsymbol{x}_0$ of $\boldsymbol{v}$ is said to be *hyperbolic* if none of the eigenvalues of A are purely imaginary.

1) Consider the vector field $v(x) = -x^3$ on $\mathbf{R}$. Show that it is not hyperbolic. However, show that 0 is an asymptotically stable equilibrium point of v by finding a strict Lyapunov function for v.

2) Consider the differential equation

$$\dot{x} = A\boldsymbol{x}$$

where A is a 2×2 matrix and consider the equilibrium point $\mathbf{0}$ for this equation. Determine if $\mathbf{0}$ is stable, asymptotically stable, or unstable for the following A

(a) $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $a < 0 < b$

(b) $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $\lambda > 0$

(c) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(d) Each of the equations above has one of the following pictures as a phase portrait. Write out which figure below corresponds to equation (a), which to (b), and which to (c) above. Note that some of the five figures below may have no equation associated with them in the list above.

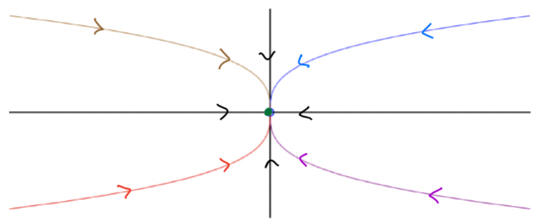

Figure 1.

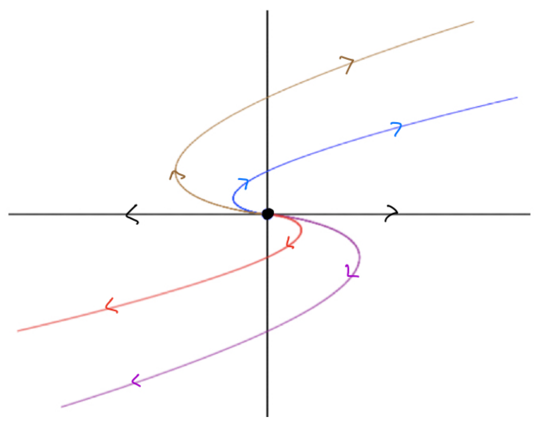

Figure 2.

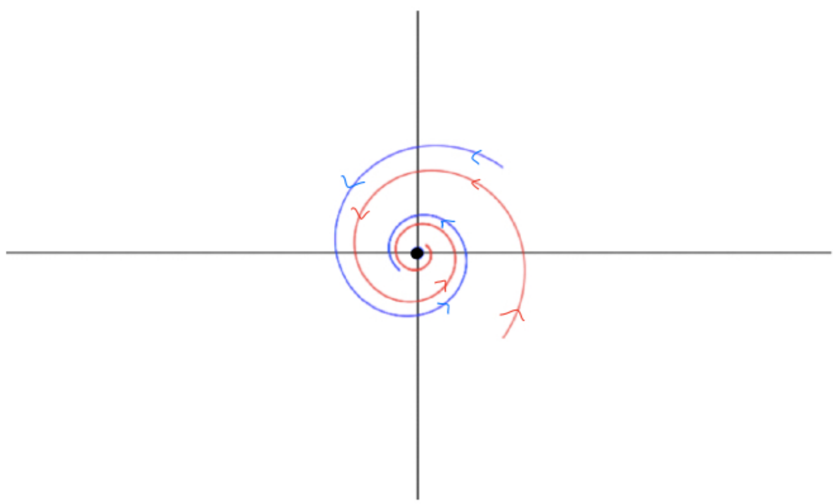

Figure 3.

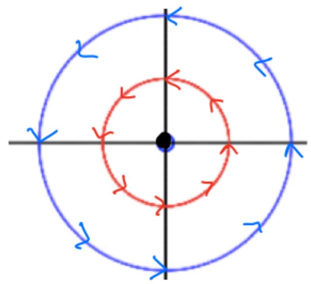

Figure 4.

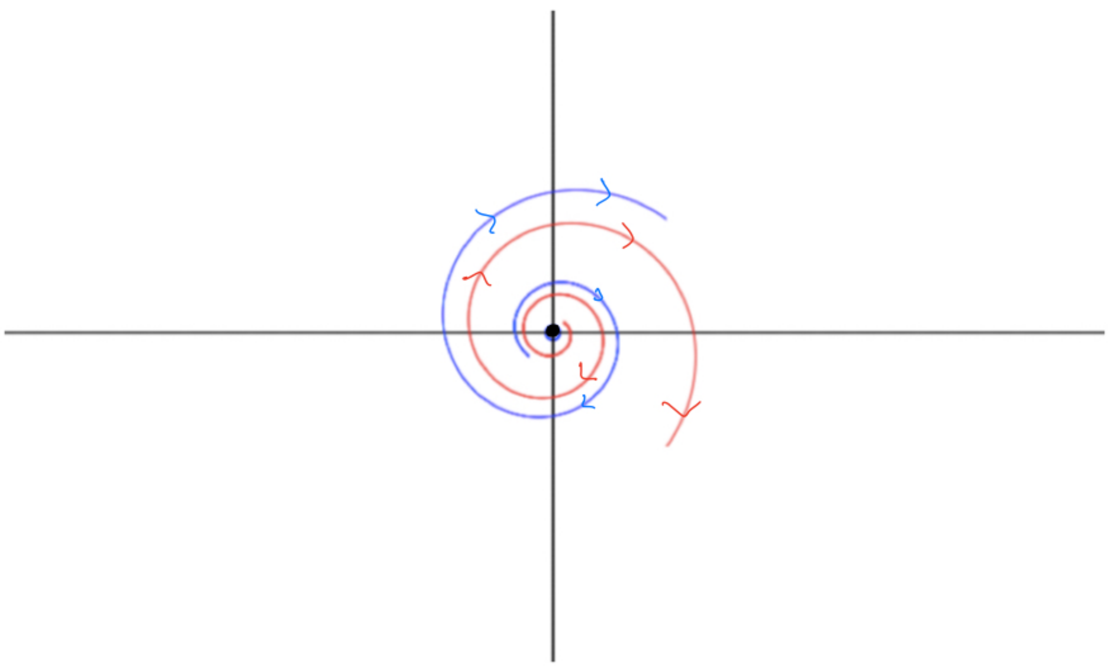

Figure 5.