QUIZ 2
FEBRUARY 8, 2021

1) Let A be an $n \times n$ matrix. Let $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ be the map which sends a vector to its last n coordinates. In other words $\pi(\boldsymbol{x}) = \boldsymbol{y}$ where $y_i = x_{i+1}$, $i = 1, \ldots, n$. Define a function $\boldsymbol{f}\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ by the formula:

$$\boldsymbol{f}(\boldsymbol{x}) = e^{x_1 A}\pi(\boldsymbol{x}).$$

Show that the $n \times (n+1)$ matrix $\boldsymbol{f}'(\boldsymbol{x})$ whose $(i,j)^{\text{th}}$ term is $\partial f_i/\partial x_j(\boldsymbol{x})$ is given in block matrix notation by

$$\boldsymbol{f}'(\boldsymbol{x}) = \left[Ae^{x_1 A}\pi(\boldsymbol{x}) \mid e^{x_1 A}\right]$$

Reparameterization. Suppose $W \subset \mathbf{R}^n$ is an open domain and $\boldsymbol{v}\colon W \to \mathbf{R}^n$ is a continuous map. Assume that for each $\boldsymbol{a} \in W$, the IVP

$$(\dagger) \qquad \begin{cases} \dot{\boldsymbol{x}} & = \boldsymbol{v}(\boldsymbol{x}) \\ \boldsymbol{x}(0) & = \boldsymbol{a} \end{cases}$$

has a solution on some time interval containing 0 in its interior and that the solution is unique on that interval. Let $h\colon W \to (0, \infty)$ be a continuous map and let $\boldsymbol{w}\colon W \to \mathbf{R}^n$ be given by

$$\boldsymbol{w}(\boldsymbol{x}) = h(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in W.$$

Assume that for each $\boldsymbol{a} \in W$, the IVP

$$(\ddagger) \qquad \begin{cases} \dot{\boldsymbol{x}} & = \boldsymbol{w}(\boldsymbol{x}) \\ \boldsymbol{x}(0) & = \boldsymbol{a} \end{cases}$$

also has a solution on some time interval containing 0 in its interior and that the solution is unique on that interval.

2) Fix $\boldsymbol{a} \in W$ and let (ω_-, ω_+) and (μ_-, μ_+) be the maximal intervals of existence for ($\dagger$) and ($\ddagger$) for this initial value $\boldsymbol{a}$ (with $t_0 = 0$), which exist by the previous set of exercises, and the fact that we are in the autonomous case and hence free to set t_0 equal to 0. Let $\boldsymbol{\varphi}\colon (\omega_-, \omega_+) \to W$ and $\boldsymbol{\psi}\colon (\mu_-, \mu_+) \to W$ be the unique solutions of ($\dagger$) and ($\ddagger$). Show that there exists an increasing function

$$j\colon (\omega_-, \omega_+) \longrightarrow (\mu_-, \mu_+)$$

such that $\boldsymbol{\psi} \circ j = \boldsymbol{\varphi}$ and $j(0) = 0$. [Hint: For each $\boldsymbol{y} \in W$ let $\boldsymbol{\varphi_y}$ and $\boldsymbol{\psi_y}$ be the solutions to $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{y}$ and $\dot{\boldsymbol{x}} = \boldsymbol{w}(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{y}$. For each t in the domain of $\boldsymbol{\varphi_y}$ let $\sigma(t, \boldsymbol{y}) = \int_0^t \frac{1}{h(\boldsymbol{\varphi_y}(u))} du$. For $\boldsymbol{y} \in W$, if t_1 is in the domain of $\boldsymbol{\varphi_y}$, say $\boldsymbol{b} = \boldsymbol{\varphi_y}(t_1)$, and t_2 is in the domain of $\boldsymbol{\varphi_b}$, $s_1 = \sigma(t_1, \boldsymbol{y})$, $s_2 = \sigma(t_2, \boldsymbol{b})$, then check that $\sigma(t_1 + t_2, \boldsymbol{y}) = s_1 + s_2$. Set j equal to the map $t \mapsto \sigma(t, \boldsymbol{a})$.]