QUIZ 1
JAN XX, 2021

Let V be a finite dimensional vector space over a field k. A linear operator $T\colon V \to V$ is called *nilpotent* if there exists a positive integer n such that $T^n = 0$. Similarly, define a square matrix A with entries in k to be nilpotent if $A^n = 0$ for some positive integer n. Clearly, once one fixes a basis for V, a linear operator T is nilpotent if and only if its associated matrix (with respect to the fixed basis) is nilpotent.

Now assume $k = \mathbf{R}$, the field of reals. Suppose T is a nilpotent linear operator on V. Define

$$e^T = \sum_{m=0}^{\infty} \frac{T^m}{m!}.$$

Note that this a finite sum since T is nilpotent. It is easy to see that

$$e^{(t+s)T} = e^{tT}e^{sT}$$

and that $\{e^{tT} \mid t \in \mathbf{R}\}$ is a one-parameter group of linear transformations on V. If A is a nilpotent matrix with entries from $\mathbf{R}$, define e^{tA} in the same way as e^{tT} is defined for a nilpotent linear operator. It is clear that if T is the linear transformation associated with A, then e^{tT} is the linear transformation associated with e^{tA}. You don't have to prove these easy assertions in this quiz, but you may use them in anything that follows.

All vector spaces, matrices, and linear transformations below are over $\mathbf{R}$.

(1) Let V be the vector space polynomials of degree less than n over $\mathbf{R}$. In other words

$$V = \{p \in \mathbf{R}[x] \mid \deg p < n\}.$$

Let $T\colon V \to V$ be the map $T = \frac{d}{dx}$. From elementary calculus, T is linear and $T^n = 0$ (there is no need to re-prove this well-known result). For $t \in \mathbf{R}$, let $H_t\colon V \to V$ be the map $H_t(p(x)) = p(x+t)$, $p(x) \in V$. Show that

$$e^{tT} = H_t.$$

(2) Let A be the $n \times n$ real upper-triangular matrix:

$$A = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}$$

Show that $A^n = 0$ and that for every $t \in \mathbf{R}$, e^{tA} is the upper triangular matrix:

$$e^{tA} = \begin{bmatrix} 1 & t & t^2/2! & \dots & t^{n-1}/(n-1)! \\ & 1 & t & \dots & t^{n-2}/(n-2)! \\ & & 1 & \dots & t^{n-3}/(n-3)! \\ & & & \ddots & t^2/2! \\ & & & & t \\ & & & & 1 \end{bmatrix}.$$

Hint: Use the first problem, by identifying V with $\mathbf{R}^n$ and choosing a suitable basis for V so that A is the matrix of the map $\frac{d}{dx} : V \to V$.