LECTURE 9

Date of Lecture: February 1, 2019

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \ldots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \, \|_2$ and we will simply denote it as $\| \, \|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\| \, \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n, \mathbf{K}^n)$.

Note that $(x_1, \ldots, x_n) \neq [x_1 \ldots x_n]$. Each side is the transpose of the other.

1. One-parameter group of transformations

1.1. Set theoretic version. Let M be a set, and

$$g \colon \mathbf{R} \times M \longrightarrow M$$

an action of the additive group $\mathbf{R}$ on M. For fixed $t \in \mathbf{R}$ write $g^t \colon M \to M$ for the map given by

$$g^t(n) = g(t, m), \qquad (m \in M).$$

We can regard g as a homomorphism

$$g \colon \mathbf{R} \to \mathrm{Aut}(M).$$

We have

$$\begin{aligned} g^t g^s &= g^{t+s} \qquad s, t \in \mathbf{R} \\ g^0 &= \mathbf{1}_M. \end{aligned} \tag{1.1.1}$$

[1]See §§2.1 of Lecture 5 of ANA2.

Definition 1.1.2. A *one-parameter group of transformations on a set M* is a map $g\colon \mathbf{R} \times M \to M$ satisfying (1.1.1). This is often denoted $\{g^t\}$ or $\{g^t \mid t \in \mathbf{R}\}$. The pair $(M, \{g^t\})$ is called a *phase flow*

There is a loose terminology associated with all this. Often g, $\{g^t\}$, $(M, \{g^t\})$ are all called phase flows. As far as information content goes, there is no difference between a one-parameter group of transformations and its phase flow. In fact g, $\{g^t\}$, $(M, \{g^t\})$ all contain exactly the same information.

Definition 1.1.3. Let $(M, \{g^t\})$ be a phase flow. Then M is called the *phase space*, or the *state space*, of the flow. A point of M is called a *phase point* or a *state*. The space $\mathbf{R} \times M$ is called the *extended phase space*. Let $x \in M$ be a phase point. The map

$$\varphi = \varphi_x \colon \mathbf{R} \longrightarrow M$$

given by

$$\varphi(t) = g^t x, \qquad t \in \mathbf{R}$$

is called the *motion* of x under the flow $(M, \{g^t\})$ and the image of φ in M is called the phase curve. The graph of φ in the extended phase space $\mathbf{R} \times M$ is called the *integral curve* of the phase flow.

Dictionary. (x a phase point)

1. Motion of x.	1. The orbit map of x.
2. Phase curve of x.	2. The orbit of x under g.

1.2. **One-parameter group of diffeomorphisms.** In what follows, feel free to replace the term “manifold” by ”open subset of Euclidean space”. By a manifold (for those willing to start thinking in those terms), we mean a smooth differential manifold. The reader is urged once again to look up the definition of a differential manifold from any source, and also look at the supplementary notes on derivations and tangent spaces.

Definition 1.2.1. By a *one-parameter group $\{g^t\}$ of diffeomorphisms of a manifold M* is meant a mapping

$$g\colon \mathbf{R} \times M \longrightarrow M, \quad g(t,x) = g^t x, \quad t \in \mathbf{R},\ x \in M$$

of $\mathbf{R} \times M$ into M such that

1. g is a $\mathscr{C}^2$ mapping;
2. The mapping $g^t\colon M \to M$ is a diffeomorphism for every $t \in \mathbf{R}$;
3. The family $\{g^t \mid t \in \mathbf{R}\}$ is a one-parameter group of transformations of M.

There is clearly a more succinct definition, namely, a one-parameter group of diffeomorphisms on a manifold M is a $\mathscr{C}^r$-map (r assumed to be ≥ 2)

$$g\colon \mathbf{R} \times M \to M$$

such that g is an action of the additive group $\mathbf{R}$ on M. Note that this forces each $g^t\colon M \to M$ to be a diffeomorphism, where $g^t(\boldsymbol{x}) = g(t, \boldsymbol{x})$ for $\boldsymbol{x} \in M$. As before, we will also use the notation $\{g^t\}$ to denote this one-parameter group of diffeomorphisms.

1.3. **The phase velocity associated with $\{g^t\}$.** Fix a one-parameter group $\{g^t\}$ of diffeomorphisms on an open set M of $\mathbf{R}^n$. For $\boldsymbol{x} \in M$, let

$$\varphi_{\boldsymbol{x}} \colon \mathbf{R} \to M$$

be the map $t \mapsto g^t \boldsymbol{x}$. As before, we write

$$\dot{\varphi}_{\boldsymbol{x}} = \frac{d\varphi_{\boldsymbol{x}}}{dt}.$$

Note that

$$\dot{\varphi}_{\boldsymbol{x}} \colon \mathbf{R} \to \mathbf{R}^n.$$

Definition 1.3.1. The *phase velocity vector* of $\{g^t\}$ at $\boldsymbol{x} \in M$ is the vector $v(\boldsymbol{x})$ in $\mathbf{R}^n$ given by the formula

$$v(\boldsymbol{x}) = \dot{\varphi}_{\boldsymbol{x}}(0) = \lim_{h \to 0} \frac{\varphi_{\boldsymbol{x}}(h) - \boldsymbol{x}}{h}.$$

The map $v \colon M \to \mathbf{R}^n$ given by $\boldsymbol{x} \mapsto v(\boldsymbol{x})$ is called the *phase velocity field.*

Theorem 1.3.2. *Let $v \colon M \to \mathbf{R}^n$ be the phase velocity field of $\{g^t\}$ and let $\boldsymbol{x_0} \in M$. Then $\varphi_{\boldsymbol{x_0}}$ is the unique solution to the autonomous IVP*

$$\dot{\boldsymbol{x}} = v(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x_0}.$$

Proof. The uniqueness follows from local uniqueness. Indeed, since g is $\mathscr{C}^2$, the phase veloctiy is $\mathscr{C}^1$. Now use the fact that $\mathscr{C}^1$ maps are locally Lipschitz to see uniqueness. We have to show that $\varphi_{\boldsymbol{x_0}}$ is a solution to the given IVP. We clearly have $\varphi_{\boldsymbol{x_0}}(0) = \boldsymbol{x_0}$. Furthermore, we have:

$$\begin{aligned}
\dot{\varphi}_{\boldsymbol{x_0}}(s) &= \lim_{h \to 0} \frac{\varphi_{\boldsymbol{x_0}}(s+h) - \varphi_{\boldsymbol{x_0}}(s)}{h} \\
&= \lim_{h \to 0} \frac{g^{s+h}\boldsymbol{x_0} - g^s\boldsymbol{x_0}}{h} \\
&= \lim_{h \to 0} \frac{g^h(g^s\boldsymbol{x_0}) - g^s\boldsymbol{x_0}}{h} \\
&= \lim_{h \to 0} \frac{\varphi_{g^s\boldsymbol{x_0}}(h) - g^s\boldsymbol{x_0}}{h} \\
&= v(g^s\boldsymbol{x_0}) \\
&= v(\varphi_{\boldsymbol{x_0}}(s)).
\end{aligned}$$

This gives the required result. □

1.3.3. Tweaks. In the definition of a one-parameter group of diffeomorphisms, we can relax the condition that g be $\mathscr{C}^2$, and merely insist that it be $\mathscr{C}^1$. In this case too, the phase velocity vector field $\boldsymbol{v}$ makes sense. On the other hand, for Theorem 1.3.2 to hold, one will need to assume $\boldsymbol{v}$ is locally Lipschitz. For that it is enough to assume (for example) that $\boldsymbol{v}$ is $\mathscr{C}^1$.

1.4. **One-parameter groups of linear transformations.**

Definition 1.4.1. A *one-parameter group of linear transformations* on $\mathbf{R}^n$ is a one-parameter group $\{g^t\}$ of diffeomorphisms on $\mathbf{R}^n$ such that each $g^t\colon \mathbf{R}^n \to \mathbf{R}^n$ is a linear transformation.

Note that a one-parameter group of linear transformations is the same as a group homomorphism

$$g\colon \mathbf{R} \to GL_n(\mathbf{R}),$$

such that g is in $\mathscr{C}^r$, where we regard $GL_n(\mathbf{R})$ as an open subset of $\mathbf{R}^{n^2}$.

Let $\{g^t\}$ be a one-parameter group of linear transformations on $\mathbf{R}^n$. Let us work out its phase velocity field. Since $g\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is C^r, therefore the map $\mathbf{R} \to GL_n(\mathbf{R}) \subset L(\mathbf{R})$, given by $t \mapsto g^t$ is C^r, and in particular has continuous derivatives. Let

$$A = \left.\frac{dg^t}{dt}\right|_{t=0}.$$

Then $A \in L(\mathbf{R}) = \mathbf{R}^{n^2}$. Note that since all norms on $\mathbf{R}^{n^2}$ are equivalent, we get

$$\lim_{h\to 0}\left\|\frac{g^h - I_n}{h} - A\right\|_{\circ} = 0$$

where $\|\cdot\|_{\circ}$ is the norm on $L(\mathbf{R}^n)$ we defined in earlier lectures.

Let I_n be the identity linear transformation on $\mathbf{R}^n$, $\|\cdot\|$ the usual Euclidean norm on $\mathbf{R}^n$. Then we have

$$\begin{aligned}\left\|\frac{g^h\boldsymbol{x} - \boldsymbol{x}}{h} - A\boldsymbol{x}\right\| &= \left\|\Big(\frac{g^h - I_n}{h} - A\Big)(\boldsymbol{x})\right\| \\ &\leq \left\|\frac{g^h - I_n}{h} - A\right\|_{\circ} \cdot \|\boldsymbol{x}\|.\end{aligned}$$

The last quantity tends to 0 as $h \to 0$. It follows that

$$\lim_{h\to 0}\frac{g^h\boldsymbol{x} - \boldsymbol{x}}{h} = A\boldsymbol{x}.$$

Thus, by definition, $A\boldsymbol{x}$ is the phase velocity at $\boldsymbol{x}$ for $\{g^t\}$.[2]

References

[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, translates by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.

[2]We prefer the symbol A over v in this case.