LECTURE 7

Date of Lecture: January 25, 2021

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \ldots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \ \|_2$ and we will simply denote it as $\| \ \|$. The space of $\mathbf{R}$-linear transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^m)$ and will be identified in the standard way with the space of $m \times n$ real matrices $M_{m,n}(\mathbf{R})$ and the operator norm[1] on both spaces will be denoted $\| \ \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$.

☡ Note that $(x_1, \ldots, x_n) \neq [x_1 \ldots x_n]$. Each side is the transpose of the other.

1. First Order Linear Equations

1.1. The main theorem we will prove today is the following.

Theorem 1.1.1. *Let $I \subset \mathbf{R}$ be an interval (closed, open, half-open), $A\colon I \to M_n(\mathbf{R})$ and $\boldsymbol{g}\colon I \to \mathbf{R}^n$ continuous maps. Let $(t_0, \boldsymbol{a_0}) \in I \times \mathbf{R}^n$. Then the IVP*

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x}(t) + \boldsymbol{g}(t) \\ \boldsymbol{x}(t_0) = \boldsymbol{a_0} \end{cases}$$

has a unique solution on I.

Proof. If I is open or half-open, we can find an increasing sequence of compact intervals $\{I_n\}$ with $t_0 \in I_n \subset I$ such that $\cup_n I_n = I$. So without loss of generality,we may assume I is compact, for the unique solutions on each I_n guarantee that they glue, and any solution of the IVP on I restricts to a solution on I_n,necessarily the unique solution on I_n.

[1]See §§2.1 of Lecture 5 of ANA2.

Suppose $I = [c, d]$ is a compact. Then

$$\boldsymbol{v} \colon I \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$$

given by

$$\boldsymbol{v}(t, \boldsymbol{x}) = A(t)\boldsymbol{x} + \boldsymbol{g}(t) \qquad ((t, \boldsymbol{x}) \in I \times \mathbf{R}^n)$$

is Lipschitz in $\boldsymbol{x}$. To see this, observe that since I is compact, and A is continuous, $\|A(t)\|_\circ$ is bounded for $t \in I$, where $\|\cdot\|_\circ$ is the operator norm on $M_n(\mathbf{R})$. Let

$$L = \sup_{t \in I} \|A(t)\|_\circ \,.$$

Then $L < \infty$ and

$$\|\boldsymbol{v}(t, \boldsymbol{x}) - \boldsymbol{v}(t, \boldsymbol{y})\| = \|A(t)(\boldsymbol{x} - \boldsymbol{y})\| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|,$$

proving the assertion. The Lipschitz constant is L.

Let

$$M = \sup_{t \in I} \|\boldsymbol{g}(t)\|.$$

Then (as $\boldsymbol{g}$ is continuous and I is compact), $M < \infty$.

For $\eta > 0$, we can extend A to $I_\eta = (c - \eta, d + \eta)$ by setting $A(t) = A(c)$ for $t \in (c - \eta, c)$ and $A(t) = A(d)$ for $t \in (d, d + \eta)$. Then A is continuous on I_η. Similarly, we may extend $\boldsymbol{g}$ in a continuous fashion to I_η, by setting $\boldsymbol{g}(t) = \boldsymbol{g}(c)$ on $(c - \eta, c)$ and $\boldsymbol{g}(t) = \boldsymbol{g}(d)$ on $(d, d + \eta)$. The bounds for $\|A(t)\|_\circ$ and $\|\boldsymbol{g}(t)\|$ remain L and M respectively on I_η. Thus, if $\boldsymbol{v}$ is extended to $I_\eta \times \mathbf{R}^n$ via the formula $\boldsymbol{v}(t, \boldsymbol{x}) = A(t)\boldsymbol{x} + \boldsymbol{g}(t)$ then it remains Lipschitz on this larger space with the same Lipschitz constant L.

The IVP can be extended to $\Omega = I_\eta \times \mathbf{R}^n$ and the extended IVP can be re-written as

$$(*) \qquad \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{v}(t,\, \boldsymbol{x}) \\ \boldsymbol{x}(t_0) = \boldsymbol{a_0} \end{cases}$$

By [Lecture 20, Thm. 1.2.1], which applies since $\boldsymbol{v}(t,\, \boldsymbol{x})$ is Lipschitz in $\boldsymbol{x}$ on Ω, there is a maximal interval of existence $(\omega_-,\, \omega_+) \subset I_\eta$ for $(*)$. We point out that since $(\omega_-,\, \omega_+) \subset I_\eta$, $\omega_+ < \infty$ and $-\infty < \omega_-$. We will show below that $(\omega_-,\, \omega_+) = I_\eta$. This will complete the proof of the theorem.

Let $\boldsymbol{\varphi}_\circ : (\omega_-,\, \omega_+) \to \Omega$ be the unique solution to the IVP $(*)$. Then

$$\boldsymbol{\varphi}_\circ(t) = \boldsymbol{a_0} + \int_{t_0}^{t} A(s)\boldsymbol{\varphi}_\circ(s)\, ds + \int_{t_0}^{t} \boldsymbol{g}(s) ds \qquad (t \in (\omega_+,\, \omega_-)).$$

Let

$$K = \|\boldsymbol{a}_0\| + M(\omega_+ - t_0).$$

Taking norms, and for simplicity picking $t \in [t_0,\, \omega_+)$, we get:

$$
\begin{aligned}
\|\boldsymbol{\varphi}_\circ(t)\| &\le \|\boldsymbol{a_0}\| + \int_{t_0}^{t} \|A(s)\boldsymbol{\varphi}_\circ(s)\|\, ds + \int_{t_0}^{t} \|\boldsymbol{g}(s)\| ds \\
&\le \|\boldsymbol{a_0}\| + \int_{t_0}^{t} \|A(s)\|_\circ \|\boldsymbol{\varphi}_\circ(s)\|\, ds + M(t-t_0) \\
&\le \|\boldsymbol{a_0}\| + L\int_{t_0}^{t} \|\boldsymbol{\varphi}_\circ(s)\|\, ds + M(t-t_0) \\
&\le \|\boldsymbol{a_0}\| + \int_{t_0}^{t} \|A(s)\|_\circ \|\boldsymbol{\varphi}_\circ(s)\|\, ds + M(t-t_0) \\
&\le \|\boldsymbol{a_0}\| + M(\omega_+ - t_0) + L\int_{t_0}^{t} \|\boldsymbol{\varphi}_\circ(s)\|\, ds \\
&= K + L\int_{t_0}^{t} \|\boldsymbol{\varphi}_\circ(s)\|\, ds
\end{aligned} \tag{$\dagger$}
$$

Let

$$f(t) = \int_{t_0}^{t} \|\boldsymbol{\varphi}_\circ(s)\|\, ds.$$

Then, for $t \in [t_0, \omega_+)$ we have $f'(t) \le K + Lf(t)$, which can be re-written as

$$f'(t) - Lf(t) \le K.$$

Multiplying both sides by the "integrating factor" e^{-Lt} we get $e^{-Lt}(f'(t) - Lf(t)) \le Ke^{-Lt}$ which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-Lt} f(t)) \le Ke^{-Lt}.$$

Integrating from t_0 to t (and noting that $f(t_0) = 0$) we get

$$
\begin{aligned}
e^{-Lt} f(t) &\le K\int_{t_0}^{t} e^{-Ls} ds \\
&= \frac{K}{L}(e^{-Lt_0} - e^{-Lt}).
\end{aligned}
$$

This yields

$$f(t) \le \frac{K}{L}\Big(e^{L(t-t_0)} - 1\Big).$$

Substitute this back in ($\dagger$) to get:

$$
\begin{aligned}
\|\boldsymbol{\varphi}_\circ(t)\| &\le K + K\Big(e^{L(t-t_0)} - 1\Big) \\
&= Ke^{L(t-t_0)} \\
&\le Ke^{L(\omega_+ - t_0)}
\end{aligned}
$$

It follows that $\|\boldsymbol{\varphi}_\circ(t)\|$ is bounded in $[t_0, \omega_+)$. By Corollary 1.1.3 of Lecture 6, this means $\omega_+ = d + \eta$. Similarly $\omega_- = c - \eta$. This completes the proof. □

We deduce immediately, as in Problem (5) of Homework 2:

Corollary 1.1.2. *If $\boldsymbol{g}(t) \equiv 0$ for $t \in I$, then the space of solutions to the underlying DE $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ in Theorem 1.1.1 is an n-dimensional vector space over $\mathbf{R}$.*

2. **Some Pictures**

Here are some pictures indicating how phase flows are affected by initial data. The planes represent $\{t\}\times M$, where M is the phase space, and $t = -1, 0, 1$. Assume we are given a continuous locally Lipschitz map $\boldsymbol{v}\colon \Omega \to \mathbf{R}^2$ where $\Omega \subset \mathbf{R} \times \mathbf{R}^2$.

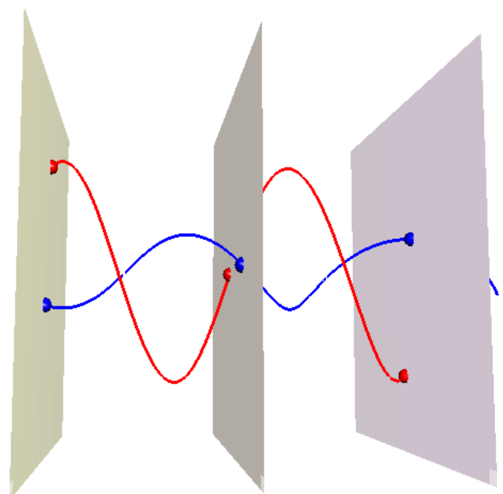

FIGURE 1. Two phase flows for the same DE with different initial conditions.

The plane on the left represents $\{-1\} \times \mathbf{R}^2$, the middle one $\{0\} \times \mathbf{R}^2$, and the one on the right $\{1\} \times \mathbf{R}^2$. The maximal interval of existence (ω_-, ω_+) contains $[-1, 1]$. Let $\boldsymbol{\xi_0} = (t_0, \boldsymbol{a_0})$ be the blue dot on the plane at $t = -1$, and $\boldsymbol{\xi_1} = (t_1, \boldsymbol{a_1})$ the red dot on the plane $t = 0$. The blue curve is the phase curve for the initial data $\boldsymbol{\xi_0}$ and the red curve the phase curve for the initial data $\boldsymbol{\xi_1}$. If we replace $\boldsymbol{\xi_0}$ with any other blue dot, we get the same phase curve (the blue one). Similarly, we may replace $\boldsymbol{\xi_1}$ with any other red dot. In fact, $\boldsymbol{\xi_1}$ may be replaced by any point on the corresponding curve passing through it.

Here are two more angles of the same phase curves (just because there is space for their pictures):

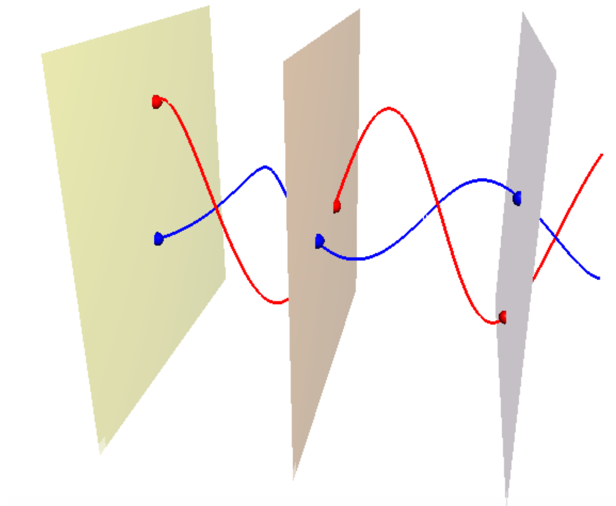

Figure 2.

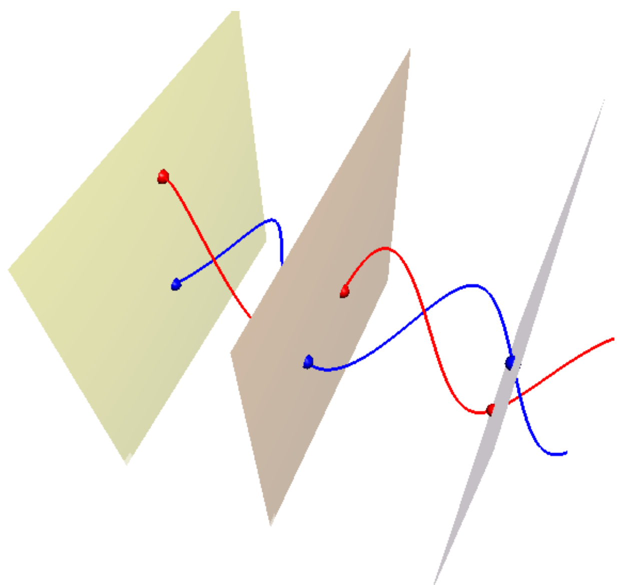

Figure 3.