LECTURE 6

Date of Lecture: January 20, 2021

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \ \|_2$ and we will simply denote it as $\| \ \|$. The space of $\mathbf{R}$-linear transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^m)$ and will be identified in the standard way with the space of $m \times n$ real matrices $M_{m,n}(\mathbf{R})$ and the operator norm[1] on both spaces will be denoted $\| \ \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. Solutions in the locally Lipschitz case

The proofs here are based on the ones given in [G].

1.1. Existence and uniqueness for locally Lipschitz $\boldsymbol{v}$.

Please refer to Lecture 5 for the definition of an interval of existence as well as that of a maximal interval of existence.

Theorem 1.1.1. *Suppose Ω is a domain in $\mathbf{R} \times \mathbf{R}^n$ and $\boldsymbol{v}\colon \Omega \to \mathbf{R}^n$ is a locally Lipschitz continous in $\boldsymbol{x}$. Fix $(t_0, \boldsymbol{a_0}) \in \Omega$. Then the IVP*

$$(*) \qquad \begin{cases} \dot{\boldsymbol{x}} &= \boldsymbol{v}(t, \boldsymbol{x}) \\ \boldsymbol{x}(t_0) &= \boldsymbol{a_0} \end{cases}$$

has a maximal interval of existence, and is of the form (ω_-, ω_+), with $\omega_- \in [-\infty, \infty)$ and $\omega_+ \in (-\infty, \infty]$. There is a unique solution

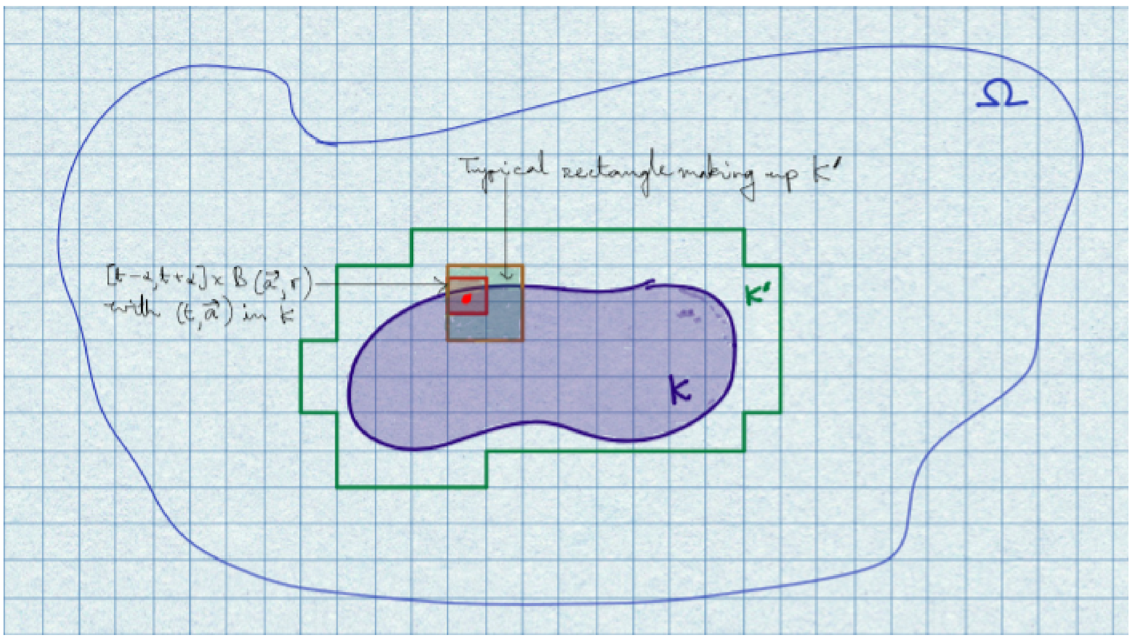
$$\boldsymbol{\varphi}_\circ = \boldsymbol{\varphi}_{(t_0, \boldsymbol{a_0})}\colon (\omega_-, \omega_+) \to \mathbf{R}^n$$

[1] See §§2.1 of Lecture 5 of ANA2.

of $(*)$ *on* (ω_-, ω_+) *and any solution of* $(*)$ *on an interval* I *containing* t_0 *is the restriction of* $\boldsymbol{\varphi}_\circ$ *to* I. *The variable point* $(t, \boldsymbol{\varphi}_\circ(t))$ *leaves every compact subset* K *of* Ω *as* $t \downarrow \omega_-$ *and as* $t \uparrow \omega_+$.

Proof. By the Picard-Lindelöf theorem, the DE $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$ has locally unique solutions for every initial data point $(\tau, \boldsymbol{a})$ in Ω and so the material covered in Lecture 5 applies. Most of the theorem has proved in the last two lectures. It remains to prove that $(t, \boldsymbol{\varphi}_\circ(t))$ leaves any given compact subset of Ω as $t \downarrow \omega_-$ and as $t \uparrow \omega_+$.

It may be useful to look at the figure below while reading the rest of the proof.



Suppose $K \subset \Omega$ is a compact subset. If $(t, \boldsymbol{a}) \in K$, we can find positive numbers $\alpha(t, \boldsymbol{a})$ and $\rho(t, \boldsymbol{a})$ such that

$$[t - 2\alpha(t, \boldsymbol{a}), t + 2\alpha(t, \boldsymbol{a})] \times \overline{B}(\boldsymbol{a}, 2\rho(t, \boldsymbol{a})) \subset \Omega$$

and such that $\boldsymbol{v}(t, \boldsymbol{x})$ is uniformly Lipschitz in $\boldsymbol{x}$ on $[t - 2\alpha(t, \boldsymbol{a}), t + 2\alpha(t, \boldsymbol{a})] \times \overline{B}(\boldsymbol{a}, 2\rho(t, \boldsymbol{a}))$. Note the factor of 2 everywhere. Since sets of the form $(t - \alpha(t, \boldsymbol{a}), t + \alpha(t, \boldsymbol{a})) \times B(\boldsymbol{a}, \rho(t, \boldsymbol{a}))$ form an open cover of K as $(t, \boldsymbol{a})$ varies in K, there exist $(t_i, \boldsymbol{a_i}) \in K$, $i = 1, \ldots, m$ such that, with $\rho_i = \rho(t_i, \boldsymbol{a_i})$ and $\alpha_i = \alpha(t_i, \boldsymbol{a_i})$ for $i = 1, \ldots, m$, we have the inclusion:

$$K \subset \bigcup_{i=1}^{m} (t_i - \alpha_i, t_i + \alpha_i) \times B(\boldsymbol{a_i}, \rho_i).$$

Let

$$K' = \bigcup_{i=1}^{m} [t_i - 2\alpha_i, t_i + 2\alpha_i] \times \overline{B}(\boldsymbol{a_i}, 2\rho_i).$$

Then K' is compact and $K' \subset \Omega$. Let $M = \sup_{(t,\boldsymbol{a})\in K'} |\boldsymbol{v}(t,\, \boldsymbol{a})|$, $\alpha = \min_{1\le i\le m} \alpha_i$, $r = \min_{1\le i\le m} \rho_i$, and

$$b = \min\Big\{\alpha, \frac{r}{M}\Big\}. \tag{†}$$

Suppose $(\tau, \boldsymbol{a}) \in K$. Then, $(\tau, \boldsymbol{a}) \in [t_i - \alpha_i, t_i + \alpha_i] \times \overline{B}(\boldsymbol{a_i}, \rho_i)$ for some $i \in \{1, \dots, m\}$. By the triangle inequality, it follows that for this i we have

$$[\tau - \alpha, \tau + \alpha] \times \overline{B}(\boldsymbol{a}, r) \subset [t_i - 2\alpha_i, t_i + 2\alpha_i] \times \overline{B}(\boldsymbol{a_i}, 2\rho_i).$$

Thus $\boldsymbol{v}(t, \boldsymbol{x})$ is uniformly Lipschitz in $\boldsymbol{x}$ on $[\tau - \alpha, \tau + \alpha] \times \overline{B}(\boldsymbol{a}, r)$ and further $[\tau - \alpha, \tau + \alpha] \times \overline{B}(\boldsymbol{a}, r) \subset K'$. By the Picard-Lindelöf theorem, we have a unique solution $\boldsymbol{\varphi}_{(\tau,\boldsymbol{a})}$ to the IVP $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$, $\boldsymbol{x}(\tau) = \boldsymbol{a}$, such that $[\tau - b, \tau + b]$ is an interval of existence, where b is as in (†). In particular $[t_0 - b, t_0 + b]$ is an interval of existence for $(*)$, which means $[t_0 - b, t_0 + b] \subset (\omega_-,\, \omega_+)$, whence $(\omega_-,\, \omega_+)$ has length strictly greater than $2b$.

Let us return to our IVP $(*)$ and its unique solution $\boldsymbol{\varphi}_\circ$ on $(\omega_-,\, \omega_+)$. If $\omega_+ = \infty$, then $\tau_{\max} := \sup\{\tau \in (\omega_-, \infty) \mid (\tau, \boldsymbol{\varphi}_\circ(\tau)) \in K\}$ is a real number in (ω_-, ∞), and $(\tau, \boldsymbol{\varphi}_\circ(\tau)) \notin K$ for $\tau > \tau_{\max}$. Similarly, if $\omega_- = -\infty$ we can find $\tau_{\min} \in (-\infty, \omega_+)$ such that $(\tau, \boldsymbol{\varphi}_\circ(\tau)) \notin K$ for $\tau < \tau_{\min}$. Now assume $\omega_+ < \infty$. By the parts of the theorem that we have already proved (see section on maximal intervals of existence in Lecture 5 as well as the Picard-Lindelöf theorem in Lecture 4) it is clear that if $\tau \in (\omega_-,\, \omega_+)$ and $\boldsymbol{a} = \boldsymbol{\varphi}_\circ(\tau)$, then $\boldsymbol{\varphi}_\circ = \boldsymbol{\varphi}_{(\tau,\boldsymbol{a})}$ and the maximal interval of existence of $\boldsymbol{\varphi}_{(\tau,\boldsymbol{a})}$ is also $(\omega_-,\, \omega_+)$. Now suppose $(\tau, \boldsymbol{\varphi}_\circ(\tau)) \in K$ for some $\tau \in (\omega_-,\, \omega_+)$. From our earlier observations, $[\tau - b, \tau + b]$ is an interval of existence for $\boldsymbol{\varphi}_{(\tau,\boldsymbol{a})}$ where $\boldsymbol{a} = \boldsymbol{\varphi}_\circ(\tau)$. This means $\omega_- < \tau - b$ and $\tau + b < \omega_+$. In other words, if if $\omega_+ - b \le \tau < \omega_+$, then $(\tau,\, \boldsymbol{\varphi}_\circ(\tau)) \notin K$. Similarly, if $\omega_- > -\infty$ then $(\tau,\, \boldsymbol{\varphi}_\circ(\tau)) \notin K$ whenever $\omega_- < \tau \le \omega_- + b$. This proves the theorem. □

1.1.2. The crucial point in the proof of the statement about compact subsets of Ω is this: Given a compact subset K of Ω, we have a positive real number $b = b_K$ such that for every $(\tau, \boldsymbol{a}) \in K$, the open interval $(\tau - b, \tau + b)$ is an interval of existence for the IVP $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$, $\boldsymbol{x}(\tau) = \boldsymbol{a}$. The number b can be chosen as in (†) above. This has the following consequence:

> *Given a point $(\tau, \boldsymbol{a}) \in \Omega$, there exists an open neighbouhood $W_{(\tau,\boldsymbol{a})}$ of $(\tau,\, \boldsymbol{a})$ in Ω and a positive number $b = b_{(\tau,\, \boldsymbol{a})}$ such that for every $(\theta,\, \boldsymbol{y}) \in W_{(\tau,\boldsymbol{a})}$ the interval $(\theta - b, \theta + b)$ is an interval of existence for the IVP $\dot{x} = \boldsymbol{v}(t,\, \boldsymbol{x})$, $\boldsymbol{x}(\theta) = \boldsymbol{y}$.*

Indeed, one can pick an open rectangle $(\tau - 2\eta, \tau + 2\eta) \times B(\boldsymbol{a}, 2r)$ which lies entirely in Ω. Then $K = [\tau - \eta, \tau + \eta] \times \overline{B}(\boldsymbol{a}, r)$ is a compact subset of Ω, and if we pick $b = b_K$ for this K (as in (†)), and $W_{(\tau,\boldsymbol{a})} = (\tau - \eta, \tau + \eta) \times B(\boldsymbol{a}, r)$, we are done. This result will be used later when we work with manifolds.

Corollary 1.1.3. *If $U \subset \mathbf{R}^n$ is a bounded set and $\Omega = (c, d) \times U$ with (c, d) an open interval in $\mathbf{R}$, then either $\omega_+ = d$ or $\boldsymbol{\varphi}_\circ(t) \to \partial U$ as $t \uparrow \omega_+$, and either $\omega_- = c$ or $\boldsymbol{\varphi}_\circ(t) \to \partial U$ as $t \downarrow \omega_-$.*

Proof. Suppose $\omega_+ \ne d$. Then $\omega_+ < d$ and $-\epsilon + \omega_+ \in (c, d)$ for sufficiently small ϵ. Let $f\colon \mathbf{R}^n \to [0, \infty)$ be the function given by the formula:

$$f(x) = \inf_{\boldsymbol{z}\in\partial U} \|\boldsymbol{x} - \boldsymbol{z}\|.$$

Since ∂U is a closed subset of $\mathbf{R}^n$, f is continuous. For $\epsilon > 0$, let

$$\Gamma_\epsilon = \{\boldsymbol{x} \in U \mid f(\boldsymbol{x}) \geq \epsilon\}$$

and

$$K_\epsilon = [-\epsilon + \omega_+,\, \omega_+] \times \Gamma_\epsilon.$$

K_ϵ is compact and for sufficiently small ϵ, K_ϵ is a non-empty subset of Ω. From the theorem, $(t, \boldsymbol{\varphi}_\circ(t))$ exits K_ϵ. It cannot exit anywhere in $\{\omega_+\} \times \Gamma_\epsilon$, for $\boldsymbol{\varphi}_\circ$ does not make sense at $t = \omega_+$. Thus there exists $\tau_\epsilon \in [-\epsilon + \omega_+,\, \omega_+)$ such that $f(\boldsymbol{\varphi}_\circ(t)) < \epsilon$ for all $t \in [\tau_\epsilon, \omega_+)$. This proves that $\boldsymbol{\varphi}_\circ(t) \to \partial U$ as $t \uparrow \omega_+$. Similarly, if $\omega_- \neq c$ then $\boldsymbol{\varphi}_\circ(t) \to \partial U$ as $t \downarrow \omega_-$. □

Corollary 1.1.4. *If $\Omega = (c, d) \times \mathbf{R}^n$, (c, d) an open interval in $\mathbf{R}$, then either $\omega_+ = d$ or $\|\boldsymbol{\varphi}_\circ(t)\| \to \infty$ as $t \uparrow \omega_+$, and either $\omega_- = c$ or $\|\boldsymbol{\varphi}_\circ(t)\| \to \infty$ as $t \downarrow \omega_-$.*

Proof. Suppose $\omega_+ \neq d$. Then $\omega_+ < d$ and for n sufficiently large, $-\frac{1}{n} + \omega_+ \in (c, d)$. This time consider

$$K_n = \left[-\frac{1}{n} + \omega_+, \omega_+\right] \times \overline{B}(\mathbf{0}, n).$$

Then K_n is compact. Repeating the argument used towards the end of the proof of the previous corollary, we can find $\tau_n \in [-\frac{1}{n} + \omega_+, \omega_+)$ such that $\|\boldsymbol{\varphi}_\circ(t)\| \geq n$ for all $t \in [\tau_n, \omega_+)$. This proves that $\|\boldsymbol{\varphi}_\circ(t)\| \to \infty$ as $t \uparrow \omega_+$. Similarly if $\omega_- \neq c$ then $\|\boldsymbol{\varphi}_\circ(t)\| \to \infty$ as $t \downarrow \omega_-$. □

References

[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, Third Edition, translated by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.

[G] C.P. Grant, *Theory of Ordinary Differential Equations*. https://www.math.utah.edu/~treiberg/GrantTodes2008.pdf, Brigham Young University.