Jan 18, 2021

Lecture 5

Let Ω be a domain $\mathbb{R}\times\mathbb{R}^n$, i.e. Ω is a connected open subset of $\mathbb{R}\times\mathbb{R}^n$. Let

$$\vec{v}: \Omega \longrightarrow \mathbb{R}^n$$

Lipschitz in the second variable.

is said to be locally Lipschitz with respect to phase if it is continuous and if for each $(t_0, \vec{a}) \in \Omega$, $\exists$ a positive number $L = L(t_0, \vec{a})$ and a product set $I \times U$ containing $(t_0, \vec{a})$ as an interior point such that for each $t \in I$, the restriction of $\vec{v}(t, -)$ to U is Lipschitz continuous with Lipschitz constant $L = L(t_0, \vec{a})$. We say it is uniformly Lipschitz (or just Lipschitz) if $L(t_0, \vec{a})$ does not depend upon $(t_0, \vec{a})$.

Maximal intervals of existence

Let Ω be a domain in $\mathbb{R}\times\mathbb{R}^n$ and $\vec{v}: \Omega \longrightarrow \mathbb{R}^n$ a continuous map such that for each $(t_0, \vec{a}) \in \Omega$ the IVP

$$(*)_{t_0,\vec{a}} \qquad \dot{\vec{x}} = \vec{v}(t, \vec{x}) \qquad \vec{x}(t_0) = \vec{a}$$

has a solution on some time interval I (open, closed, half open) containing t_0 in its interior, and the solution is unique on this interval.

Let us fix $(t_0, \vec{a}) \in \Omega$. If I is an interval on which a solution to $(*)_{t_0,\vec{a}}$ exists, with t_0 in the interior of I, we call I an interval of existence for $(*)_{t_0,\vec{a}}$.

Suppose I_1, I_2 are open intervals of existence. Let $\vec{\Phi}_1$ and $\vec{\Phi}_2$ be solutions of $(*)_{t_0,\vec{a}}$ in I_1, I_2 respectively. From the hypothesis, the set

$$S = \{t \in I_1 \cap I_2 \mid \vec{\Phi}_1(t) = \vec{\Phi}_2(t)\}$$

is open. Indeed if $\tau \in S$, and $\vec{a}^* = \vec{\Phi}_1(\tau) = \vec{\Phi}_2(\tau)$, then $\vec{\Phi}_1$ and $\vec{\Phi}_2$ are solutions of $(*)_{\tau, a^*}$, so in a neighbourhood of τ, $\vec{\Phi}_1$ and $\vec{\Phi}_2$ agree. On the other hand S is clearly closed in $I_1 \cap I_2$ and non-empty, since $t_0 \in S$. Since $I_1 \cap I_2$ is connected, this means $S = I_1 \cap I_2$.

Conclusion: $\vec{\Phi}_1$ and $\vec{\Phi}_2$ agree on $I_1 \cap I_2$.

From the above, the union of all open intervals of existence for $(*)_{t_0,\vec{a}}$ is also an interval of existence. Let

$$J_{max} = (\omega_-, \omega_+) := \bigcup I$$

where the union is taken over all open intervals of existence. Then J_{max} is an interval of existence.

Now suppose I is an interval of existence of $(*)_{t_0,\vec{a}}$. If I is open, clearly $I \subseteq J_{max}$. A little thought shows that I is contained in J_{max} even if I is not open. This is seen as followed. Suppose $I = (a, b]$ (other cases can be dealt with in a similar way). Now $I^\circ = (a,b) \subset (\omega_-, \omega_+)$. If $I \not\subset (\omega_-, \omega_+)$ then clearly $b = \omega_+$ (since $(a,b) \subset (\omega_-, \omega_+)$). Let $\vec{\varphi} : I \longrightarrow \Omega$ be a soln of our IVP. Then $(b, \vec{\varphi}(b)) \in \Omega$, and since $\vec{v}$ is locally Lipschitz in the second variable there is a compact rectangle $R' = [b-2\alpha, b+2\alpha] \times \overline{B}(\vec{\varphi}(b), 2r) \subset \Omega$ such that $\vec{v}$ is uniformly Lipschitz in the second variable on R'. Let

$$R = [b-\alpha, b+\alpha] \times \overline{B}(\vec{\varphi}(b), r) \subset R'.$$

Let M be the supremum of $\vec{v}(t,\vec{x})$ on R'. If $(\tau, \vec{w}) \in R$, then clearly $R(\tau,\vec{w}) := [\tau-\alpha, \tau+\alpha] \times \overline{B}(\vec{w}, r) \subset R'$ (by the triangle inequality). Moreover, $\vec{v}$ is uniformly Lipschitz in the second variable on $R(\tau,\vec{w})$. Set $\beta = \min(\alpha, \frac{M}{r})$. By Picard-Lindelöf on $R(\tau,\vec{w})$ we see that for each $(\tau,\vec{w}) \in R$, the IVP $\dot{\vec{x}} = \vec{v}(t,\vec{x})$, $\vec{x}(\tau) = \vec{w}$ has a unique solution on $[\tau-\beta, \tau+\beta]$. Since $(t, \vec{\varphi}(t)) \longrightarrow (b, \vec{\varphi}(b))$ as $t \uparrow b$, we can find $\tau \in (b-\frac{\beta}{2}, b)$ such that $(\tau, \varphi(\tau)) \in R$. Let $\vec{w} = \varphi(\tau)$.

Now (i) the soln $\vec{\psi}$ of the IVP $\dot{\vec{x}} = \vec{v}(t,\vec{x})$, $\vec{x}(\tau) = \vec{w}$ has to agree with $\vec{\varphi}$ on $(b-\frac{\beta}{2}, b)$ by uniqueness of solns, and (ii) $\vec{\psi}$ exists on $[\tau-\beta, \tau+\beta]$. Since $b - \frac{\beta}{2} < \tau < b$, clearly $\tau+\beta > b$. Thus $\vec{\varphi}$ exists on $(a, \cancel{b+\beta}\ \tau+\beta)$. But $b = \omega_+$ (see argument above). This gives a contradiction since $\tau+\beta > b = \omega_+$. Hence $I \subset J_{max}$. We thus have

Theorem: There is a maximal solution $\vec{\varphi}_{max} : J_{max} = (\omega_-, \omega_+) \longrightarrow \Omega$ of the IVP $(*)_{(t_0, \vec{x}_0)}$ in the sense that if $\vec{\varphi} : I \longrightarrow \Omega$ is any soln of $(*)_{(t_0,\vec{x}_0)}$, with $t_0 \in I$, then $I \subset J_{max}$ and $\vec{\varphi} = \vec{\varphi}_{max}|_I$.

Remark For us, intervals have non-empty interiors.

Return to the one-variable autonomous case:

Consider again the IVP

$$\dot{x} = v(x), \qquad x(t_0) = x_0$$

Here $v : \Omega \longrightarrow \mathbb{R}$ is $\mathscr{C}^1$ map on an interval Ω, and $v(x_0) \neq 0$. Let (x_m, x_M) be the connected component of Ω^{reg} containing t_0. Let $(-\omega_-, \omega_+)$, J_{max}, θ, φ_{max} etc be as before. Recall, if $v(x_0) > 0$

$$\lim_{t \to \omega_+} \varphi_{max}(t) = x_M, \quad \lim_{t \to \omega_-} \varphi_{max}(t) = x_m$$

Further (again assuming $v(x_0) > 0$),

$$x_M \in \Omega \Rightarrow \omega_+ = \infty$$

$$x_m \in \Omega \Rightarrow \omega_- = -\infty.$$

Suppose K is a compact set in $J_{max} \times \Omega$. Without loss of generality can assume $K = J \times S$, with J and S closed intervals, J containing t_0.

If $x_M \in S$, then $\varphi(t)$ can never be x_M, and $\omega_+ = \infty$. In this case $(t, \varphi(t))$ exits K from the right at some point. Let $S = [x_1, x_2]$. If $x_M \notin K$, then $x_2 < x_M$. Know that $\lim_{t \to \omega_+} \varphi(t) = x_M$. Hence either $(t, \varphi(t))$ exits K from the top or from the right side.

The argument (using time reversals if necessary) shows that $(t, \varphi_{max}(t))$ must exit K.

Theorem: Suppose Ω is a domain in $\mathbb{R}\times\mathbb{R}^n$ and $\vec{v}: \Omega \longrightarrow \mathbb{R}^n$ a locally Lipschitz in $\vec{x}$ & continuous function. Let $(t_0, \vec{x_0}) \in \Omega$. Then the IVP

$$(*)_{t_0,\vec{a_0}} \quad \begin{cases} \dot{\vec{x}} = \vec{v}(t, \vec{x}) \\ \vec{x}(t_0) = \vec{a_0} \end{cases}$$

has a maximal interval of existence, and is of the form (ω_-, ω_+), $\omega_- \in [-\infty, \infty)$ and $\omega_+ \in (-\infty, \infty]$. There is a unique solution

$$\vec{\varphi_0} = \vec{\varphi}_{(t_0, \vec{a_0})} : (\omega_-, \omega_+) \longrightarrow \mathbb{R}^n$$

of $(*)_{t_0,\vec{x}_0}$ on (ω_-, ω_+) and any solution of $(*)_{t_0,\vec{a}_0}$ on an interval I containing t_0 is the restriction of $\vec{\varphi_0}$ to I. The variable point $(t, \varphi_0(t))$ leaves every compact subset K of Ω as $t \downarrow \omega_-$ and as $t \uparrow \omega_+$.

Proof:

We only need to prove the last statement. Everything else has been proved.

Let $K \subset \Omega$ be a compact subset. If $(t, \vec{a}) \in K$, we can find +ve numbers $\alpha(t, \vec{a})$ and $\rho(t, \vec{a})$ such that

$$[\, t - 2\alpha(t, \vec{a}),\ t + 2\alpha(t, \vec{a})] \times \bar{B}(\vec{a}, 2\rho(t, \vec{a})) \subset \Omega.$$

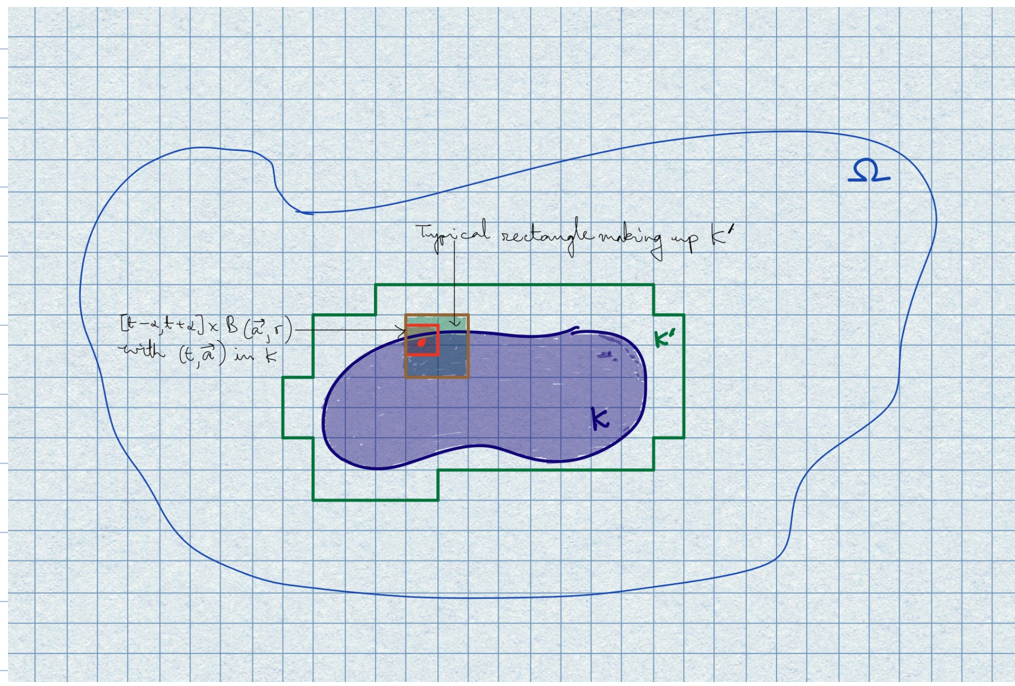
and such that $v(t, \vec{x})$ is uniformly Lipschitz on $[t - 2\alpha(t, \vec{a}),\ t + 2\alpha(t, \vec{a})] \times \bar{B}(\vec{a}, 2\rho(t, \vec{a}))$. Note the factor of 2 everywhere. Since the sets of the form $(t - 2\alpha(t, \vec{a}),\ t + 2\alpha(t, \vec{a})) \times B(\vec{a}, 2\rho(t, \vec{a}))$ form an open cover of K as $(t, \vec{a})$ varies in K, there exist $(t_i, \vec{a}_i) \in K$, $i = 1, \dots, m$ such that

$$K \subset \bigcup_{i=1}^{m} (t - \alpha_i, t + \alpha_0) \times B(a_i, \rho_i)$$

where $\alpha_i = \alpha(t_i, \vec{a}_i)$

$\leftarrow$ $\rho_i = \rho(t_i, \vec{a}_i)$

Let $\quad K' = \bigcup_{i=1}^{m} [t - 2\alpha_i, t + 2\alpha_i] \times \bar{B}(a_i, 2\rho_i)$



Then K' is compact and $K' \subset \Omega$. Let

$$M = \sup_{(t, \vec{a}) \in K'} \|\vec{v}(t, \vec{a})\|,$$

$$\alpha = \min_{1 \le i \le m} \alpha_i$$

$$r = \min_{1 \le i \le m} \rho_i.$$

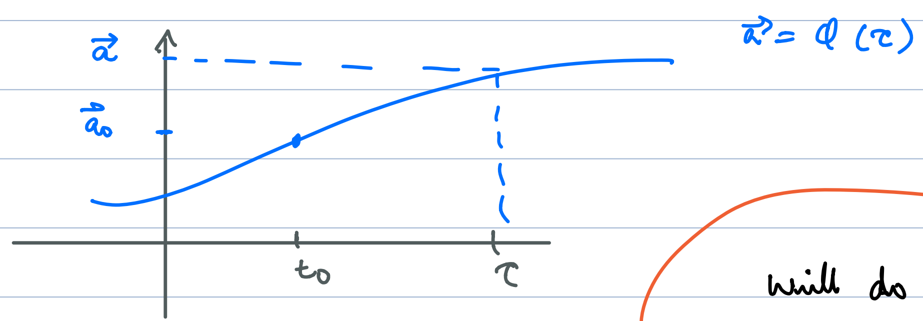
Suppose $(\tau, \vec{a}) \in K$. Then $(\tau, \vec{a}) \in [t_i - \alpha_i, t_i + \alpha_i] \times \bar{B}(\vec{a}_i, \rho_i)$ for some $i \in \{1, \dots, m\}$. By the triangle inequality

$$[\tau - \alpha, \tau + \alpha] \times \bar{B}(\vec{a}, r) \subset [t_i - 2\alpha_i, t_i + 2\alpha_i] \times \bar{B}(a_i, 2\rho_i)$$

Then v is uniformly Lipschitz in $\vec{x}$ on $[\tau - \alpha, \tau + \alpha] \times \bar{B}(\vec{a}, r)$, and further $[\tau - \alpha, \tau + \alpha] \times \bar{B}(a, r) \subset K'$. By Picard-Lindelöf, we have a solution $\varphi_{(\tau, \vec{a})}$ to the IVP $(*)_{(\tau, \vec{a})}$, $\dot{\vec{x}} = \vec{v}(t, \vec{x})$, $x(\tau) = \vec{a}$, on $[\tau - b, \tau + b]$ where

$$b = \min\left\{\alpha, \frac{r}{M}\right\} \longleftarrow \text{independent of } (\tau, \vec{a})$$

In particular $[t_0 - b, t_0 + b]$ is an interval of existence for $\vec{\varphi}_0$. Let (ω_-, ω_+) be the maximal interval of existence of $(*)_{(t_0, \vec{a}_0)}$, and $\vec{\varphi}_0$ the soln on this interval. Let $\tau \in (\omega_-, \omega_+)$, and $\vec{a} = \vec{\varphi}_0(\tau)$. Then $\vec{\varphi}_{(\tau, \vec{a})} = \vec{\varphi}_0$, and the maximal interval of existence of $(*)_{(\tau, \vec{a})}$ is also (ω_-, ω_+)



Will do the rest next time.