Jan 13, 2021

Lecture 4

Last time we started proving:

Proposition: Let $\varphi : (a,b) \longrightarrow \Omega$ be a solution of $(*)$ with $x_0 \in \Omega^{reg}$. Let θ, φ_{max}, x_m, x_n, ω_-, ω_+ etc be as above. Then

(a) φ takes values in Ω^{reg}

(b) $(a,b) \subseteq (\omega_-, \omega_+)$

(c) $\varphi = \varphi_{max}\big|_{(a,b)}$

Proof:

In the last lecture we showed that (a) implies (b) and (c). So we only have to prove (a).

Let (α, β) be the connected component of $\varphi^{-1}(\Omega^{reg})$ containing t_0.

Now $\varphi\big|_{(\alpha,\beta)}$ takes values in Ω^{reg}. From a Prop^n proved in the last lecture, it follows that

$$(\alpha, \beta) \subseteq (\omega_-, \omega_+)$$

and $$\varphi(s) = \varphi_{max}(s) \qquad \forall\, s \in (\alpha, \beta).$$

It is enough for us to show that $a = \alpha$ and $b = \beta$.

Here is the situation:

Would like to say this is an equality.

$$(a,b) \subseteq (\alpha,\beta) \subseteq (\omega_-, \omega_+)$$

WLOG enough to assume $v(x_0) > 0$.

We will prove $b = \beta$. The same proof mutatis mutandis will show that $a = \alpha$.

Suppose $\beta < b$ and $\beta < \omega_+$. Now, by taking limits, clearly $\phi(\beta) = \phi_{max}(\beta) \in \Omega^{reg}$. This contradicts the fact that (α, β) is the largest open interval containing t_0 in $\phi^{-1}(\Omega^{reg})$. Therefore either $\beta = b$ or $\beta = \omega_+$.

Next suppose $\beta = \omega_+$ and $\beta < b$. Since ϕ is defined on (a, b), $\phi(\beta)$ makes sense. Clearly $v(\phi(\beta)) = 0$. Moreover ϕ is increasing in (α, β) because v is nowhere vanishing on (α, β) and $v(\phi(t_0)) = v(x_0) > 0$. A little thought shows that

$$\phi(\beta) = x_M.$$

This means $x_M \in \Omega$, which then implies that $\omega_+ = \infty$, and hence $\beta = \omega_+ = \infty$ and $\beta < b$. This is also a contradiction.

So if $\beta = \omega_+$ then $\beta = b$.

The same argument shows that $\alpha = a$.

q.e.d.

$\longrightarrow$ P.T.O.

Theorem: Consider the IVP

(*) $\qquad \dot{x} = v(x) \qquad x(t_0) = x_0$

where $v : \Omega \longrightarrow \mathbb{R}$ is a $\mathcal{C}^1$ function and Ω is an open interval in $\mathbb{R}$. Then there a maximal solution (J_{max}, φ_{max}) of (*) such that if (J, φ) is a soln of (*) then $J \subseteq J_{max}$ and $\varphi = \varphi_{max}|_J$.

Proof:

Suppose $v(x_0) \neq 0$. Then we have already seen this is true. Suppose $v(x_0) = 0$. We claim that the max'l solution of (*) is the constant function

$$\varphi(t) \equiv x_0 \qquad t \in \mathbb{R}.$$

(In this case $J_{max} = \mathbb{R}$)

It is clear that the constant function defined above is a solution of (*). Conversely suppose (J, ψ) is a solution of (*). Suppose $v(\psi(\tau)) \neq 0$ for some $\tau \in J$. Then $\psi(t) \in \Omega^{reg}$ in a neighbourhood of τ. Let $x^* = \varphi(\tau)$. Then ψ is a soln of the IVP

$$\dot{x} = v(x) \qquad x(\tau) = x^*.$$

From the previous Propn, ψ takes values only in Ω^{reg}. This is a contradiction.

q.e.d.

Definition: Let $A \subseteq \mathbb{R}^n$. A function $\vec{f} : A \to \mathbb{R}^n$ is said to be Lipschitz if $\exists\ L > 0$ such that
$$\|\vec{f}(\vec{x}) - \vec{f}(\vec{y})\| \leq L \|\vec{x} - \vec{y}\| \qquad \forall\ \vec{x}, \vec{y} \in A.$$
The constant L is called a Lipschitz constant for $\vec{f}$.

Suppose $I = [a, b)$, $a \in (-\infty, \infty)$, $b \in (a, \infty]$. We say $\vec{f} : I \to \mathbb{R}^n$ is $\mathcal{C}^1$ if $\left.\frac{d^+}{dt}\right|_{t=a} \vec{f}$ exists, and the resulting function $\dot{\vec{f}} : I \longrightarrow \mathbb{R}^n$ is continuous. Here $\dot{\vec{f}}(a)$ is the one-sided derivative we just mentioned.

Similarly we can make sense of $\mathcal{C}^1$ functions on $(a, b]$, $[a, b]$.

Notations:

Let $\vec{a} \in \mathbb{R}^n$, $r > 0$.
$$B(\vec{a}, r) = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{a}\| < r\}$$
$$\overline{B}(\vec{a}, r) = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{a}\| \leq r\}$$

Let I be a closed and bounded interval in $\mathbb{R}$.

$C(I, n)$ = The set of continuous functions on I taking values in $\mathbb{R}^n$.

We know from earlier courses that $(C(I, n), \|\cdot\|_\infty)$ is a Banach space, i.e. it is a complete normed vector space.

To prove the existence and !-ness of solutions to DE's to IVP's on I, it is more convenient to work with a different norm denoted $\| \ \|_w$.

Let t_0 be the mid point point of I (recall I is now closed & bdd), and $|I| = 2b$. Then $I = [t_0 - b, t_0 + b]$. Let L be a positive number.

Define

$$\|\cdot\|_w = \|\cdot\|_{w,L} : C(I, \mathbb{R}^n) \longrightarrow [0, \infty)$$

by

$$\|\vec{f}\|_w = \sup_{t \in I} \left\{ e^{-2L|t-t_0|} \, |\vec{f}(t)| \right\}$$

It is straightforward to check that $\|\cdot\|_w$ is a norm.

Lemma: $\|\cdot\|_w$ and $\|\cdot\|_\infty$ are equivalent norms.

Proof:

$$e^{-Lb} \cdot \|\vec{f}\|_\infty \leq \|\vec{f}\|_w \leq \|\vec{f}\|_\infty \qquad \forall \vec{f} \in C(I, \mathbb{R}^n).$$

q.e.d.

$\longrightarrow$ P.T.O.

Theorem (Picard-Lindelöf): Let $\vec{a} \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$ and let

$$\vec{v} : [t_0-\alpha, t_0+\alpha] \times \bar{B}(\vec{a}, r) \longrightarrow \mathbb{R}^n$$

be a continuous map with upper bound M for $\|v(t,\vec{x})\|$. Suppose further that there is a positive constant L such that for each $t \in [t_0-\alpha, t_0+\alpha]$, the function $\vec{v}(t,-) : \bar{B}(\vec{a}, r) \longrightarrow \mathbb{R}^n$ is Lipschitz with Lipschitz constant L. Then the IVP

$$\dot{\vec{x}} = \vec{v}(t, \vec{x}) \quad , \quad \vec{x}(t_0) = \vec{a}$$

has a unique solution defined on $[t_0-b, t_0+b]$ where $b = \min\{\alpha, \frac{r}{M}\}$.

Proof:

Let $I = [t_0-b, t_0+b]$ where $b = \min\{\alpha, \frac{r}{M}\}$. Let $\|\cdot\|_w = \|\cdot\|_{w,L}$ where L is the constant in the statement of the theorem. Recall

$$\|\vec{f}\|_w = \sup_{t \in I} \{ e^{-2L|t-t_0|} \, |\vec{f}(t)| \}$$

Let $X = \{ \vec{f} \in C(I, n) \mid \vec{f}(I) \subseteq \bar{B}(\vec{a}, r) \}$

$$= \{ \vec{f} \in C(I,n) \mid \|\vec{f}(t) - \vec{a}\| \leq r, \ t \in I \}$$

Clearly, X is closed in $C(I,n)$. Therefore $(X, \|\cdot\|_\infty)$ is complete. Since $\|\cdot\|_\infty$ and $\|\cdot\|_w$ are equivalent, $(X, \|\cdot\|_w)$ is also complete.

P.T.O $\rightarrow$.

For $\vec{f} \in X$ and $t \in I$, define

$$(T\vec{f})(t) := \vec{a} + \int_{t_0}^{t} \vec{v}(s, \vec{f}(s))\, ds.$$

The map $t \longmapsto (T\vec{f})(t)$ is diff'ble on I (fund'l thm of calc.) and hence continuous. Now

$$\|T\vec{f} - \vec{a}\| = \left\| \int_{t_0}^{t} \vec{v}(s, \vec{f}(s))\, ds \right\| \leq \left| \int_{t_0}^{t} \|\vec{v}(s, \vec{f}(s))\|\, ds \right|$$

$$\leq M \left| \int_{t_0}^{t} ds \right|$$

$$\leq M |t - t_0|$$

$$\leq Mb$$

$$\leq r \quad \text{since } b = \min\{a, \tfrac{r}{M}\} \text{ hence } b \leq \tfrac{r}{M}$$

Thus $T\vec{f} \in X$. Hence we have a map

$$T : X \longrightarrow X.$$

Suppose $\vec{f}, \vec{g} \in X$. Then:

$$e^{-2L|t-t_0|} \cdot \left| (T\vec{f})(t) - (T\vec{g})(t) \right|$$

$$= e^{-2L|t-t_0|} \left| \int_{t_0}^{t} \{\vec{v}(s, \vec{f}(s)) - \vec{v}(s, \vec{g}(s))\}\, ds \right|$$

$$\leq e^{-2L|t-t_0|} \cdot \left| \int_{t_0}^{t} \|\vec{v}(s, \vec{f}(s)) - \vec{v}(s, \vec{g}(s))\|\, ds \right|$$

$$\leq e^{-2L|t-t_0|} \cdot L \left| \int_{t_0}^{t} \|\vec{f}(s) - \vec{g}(s)\|\, ds \right|$$

.

$$\leq L\, e^{-2L|t-t_0|} \left| \int_{t_0}^{t} e^{2L|s-t_0|} \|\vec{f}-\vec{g}\|_w \, ds \right|$$

$$= \frac{L \|\vec{f}-\vec{g}\|_w \, e^{-2L|t-t_0|}}{2L} \left| e^{2L|t-t_0|} - 1 \right|$$

$$= \frac{1}{2} \|\vec{f}-\vec{g}\|_w \left(1 - e^{-2L|t-t_0|}\right)$$

$$\leq \frac{1}{2} \|\vec{f}-\vec{g}\|_w$$

Thus

$$\|T\vec{f} - T\vec{g}\|_w \leq \frac{1}{2} \|\vec{f}-\vec{g}\|_w .$$

Hence T is a contraction map with contraction factor $\frac{1}{2}$.
It follows that T has a unique fixed point, say $\vec{\varphi}$.
This translates to

$$\vec{\varphi}(t) = \vec{a} + \int_{t_0}^{t} \vec{v}(s, \vec{\varphi}(s))\, ds,$$

whence $\vec{\varphi}$ is a solution of the given IVP.

Conversely if Ψ is a solution of the given IVP in $[t_0-b, t_0+b]$, then by the fundamental thm of calc., for $t \in I$

$$\left.\begin{aligned} \vec{\Psi}(t) &= \vec{a} + \int_{t_0}^{t} \dot{\vec{\Psi}}(s)\, ds \\ &= \vec{a} + \int_{t_0}^{t} \vec{v}(s, \vec{\Psi}(s))\, ds \end{aligned}\right\} (+)$$

Hence, for $t \in I$

$$\begin{aligned} |\vec{\Psi}(t) - \vec{a}| &= \left| \int_{t_0}^{t} \vec{v}(s, \vec{\Psi}(s))\, ds \right| \\ &\leq M \cdot |t-t_0| \\ &\leq Mb \leq r. \quad \text{(as before)}. \end{aligned}$$

No need as Arka pointed out. This is already a soln.

Thus $\vec{\Psi} \in X$. From (+) $T\vec{\Psi} = \vec{\Psi}$, and hence $\vec{\Psi} = \vec{\varphi}$.

q.e.d.