Jan 11, 2021

Lecture 3

Last time :

(*) ———— $\dot{x} = v(x), \quad x(t_0) = x_0.$

$\Omega \subseteq \mathbb{R}$, Ω is an interval.

$S_{max} = (x_m, x_M)$ = the connected component of Ω^{reg} containing x_0, where $\Omega^{reg} = \{x \in \Omega \mid v(x) \neq 0\}$.

Define
$$\theta : (x_m, x_M) \longrightarrow \mathbb{R}$$
by
$$\theta(x) = t_0 + \int_{x_0}^{x} \frac{1}{v(\xi)}\, d\xi$$

This is 1-1 C^2, has C^2 inverse φ.

Let $J_{max} = (\omega_-, \omega_+) := \theta\{(x_m, x_M)\}$.

Have $\theta : \underset{S_{max}}{\underset{\parallel}{(x_m, x_M)}} \longrightarrow \underset{J_{max}}{\underset{\parallel}{(\omega_-, \omega_+)}}$

This has a C^2 inverse

$$\varphi_{max} : (\omega_-, \omega_+) \longrightarrow (x_m, x_M).$$

We saw that φ_{max} is a soln of our IVP.

Now suppose $\varphi : (a, b) \longrightarrow \Omega^{reg}$ is a solution of (*). Note that we are requiring φ to take values in Ω^{reg}. We will show later that every soln of (*) takes values in Ω^{reg}. Let $J = (a, b)$. Now $\varphi(J)$ is connected and contains x_0, and hence, $\varphi(J) \subseteq (x_m, x_M)$ $= S_{max}$, since $S_{max} = (x_m, x_M)$ is the largest interval in Ω^{reg} containing x_0. Now $v(\varphi(s)) \neq 0$ for $s \in (a, b) = J$.

Have $\dot{\varphi}(s) = v(\varphi(s))$, $s \in (a,b)$. Hence

$$\frac{\dot{\varphi}(s)}{v(\varphi(s))} = 1 \qquad \forall s \in (a,b)$$

Integrating both sides from t_0 to t for $t \in (a,b)$ we get

$$\int_{t_0}^{t} \frac{\dot{\varphi}(s)\, ds}{v(\varphi(s))} = \int_{t_0}^{t} ds = t - t_0. \quad , t \in J$$

Use the substitution $\xi = \varphi(s)$. Get

$$\int_{x_0}^{\varphi(t)} \frac{d\xi}{v(\xi)} = t - t_0. \qquad , t \in J$$

$$\Rightarrow \qquad t_0 + \int_{x_0}^{\varphi(t)} \frac{d\xi}{v(\xi)} = t \qquad t \in J.$$

i.e. $\qquad \theta(\varphi(t)) = t.$

It is immediate that $J \subseteq (\omega_-, \omega_+)$ since the image of θ is $(-\omega_-, \omega_+) = S_{max}$. Note that $\dot{\varphi} = v \circ \varphi$ which is nowhere vanishing, and hence φ is also 1-1. This implies that φ is an inverse of $\theta|_{\varphi(J)}$.
Let $S = \varphi(J)$. We then have

$$J = (a,b) \subseteq (-\omega_-, \omega_+) = S_{max}, \quad \varphi^{-1} = \theta|_S.$$

It follows that

$$\varphi = \varphi_{max}|_{(a,b)}.$$

To summarize, if $\varphi : (a,b) \longrightarrow \Omega^{reg}$ is a solution of (*) then

P.T.O. $\rightarrow$

$$(**)\quad \begin{cases} \bullet \quad (a,b) \subseteq (\omega_-, \omega_+) \\ \bullet \quad \varphi = \varphi_{max}\Big|_{(a,b)}. \end{cases}$$

The statement $(**)$ explains the notation φ_{max}.

A stronger statement can be made, namely, if $\varphi : (a,b) \longrightarrow \Omega$ is a soln $(*)$, then φ must take values in $\Omega^{+,-}$, whence $(**)$ has to be true for φ.

Will prove this later.

Some observations:

θ and φ_{max} being monotone and continuous, are homeomorphisms between (x_m, x_M) and (ω_-, ω_+). If $v(x_0) > 0$, they are both strictly increasing, and if $v(x_0) < 0$ they are both strictly decreasing.

This gives the following:-

(1) If $v(x_0) > 0$ then

$$\lim_{x \to x_m} \theta(x) = \omega_-, \quad \lim_{x \to x_M} \theta(x) = \omega_+$$

and

$$\lim_{t \to \omega_-} \varphi_{max}(t) = x_m, \quad \lim_{t \to \omega_+} \varphi_{max}(t) = x_M.$$

$\longrightarrow$ P.T.O.

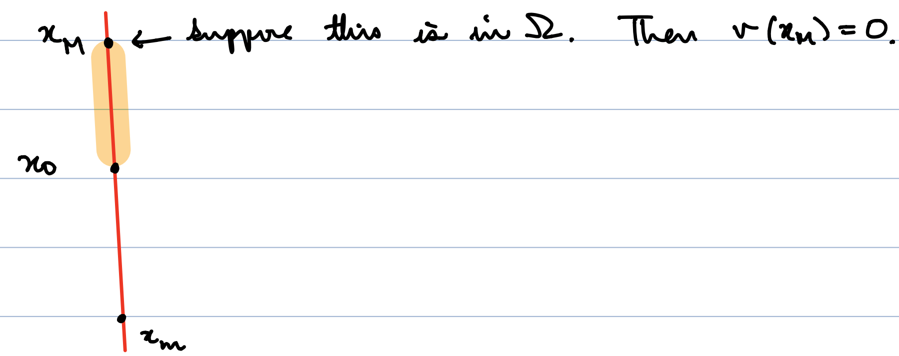
(ii) If $v(x_0) < 0$, then

$$\lim_{x \to x_m} \theta(x) = \omega_+ \,, \quad \lim_{x \to x_M} \theta(x) = \omega_-$$

and

$$\lim_{t \to \omega_+} \varphi_{max}(t) = x_m, \quad \lim_{t \to \omega_-} \varphi_{max}(t) = x_M .$$

Suppose, for the sake of definiteness, $v(x_0) > 0$, so that (i) applies. We claim that if $x_M \in \Omega$ then $\omega_+ = \infty$.



Next let

$$M = \sup \{ \dot{v}(\xi) \mid \xi \in [x_0, x_M] \}$$

Note $0 < M < \infty$. By the mean value theorem, for each $x \in [x_0, x_M)$ there exists $x^* \in [x, x_M)$ s.t.

$$v(x) = v(x) - v(x_M) = \dot{v}(x^*)(x - x_M) \qquad (\because v(x_M) = 0)$$

whence

$$v(x) = |v(x)| \leq M(x_M - x) \,, \quad x \in [x_0, x_M)$$

This means

$$\frac{1}{v(\xi)} \geq \frac{1}{M} \cdot \frac{1}{x_M - \xi} \,, \quad \xi \in [x_0, x_M)$$

Hence, for $x \in [x_0, x_M)$

$$\theta(x) = t_0 + \int_{x_0}^{x} \frac{d\xi}{v(\xi)} \geq t_0 + \frac{1}{M}\int_{x_0}^{x} \frac{d\xi}{x_M - \xi}$$

$$= t_0 + \frac{1}{M} \log \frac{x_M - x_0}{x_M - x}$$

Thus $\omega_+ = \lim_{x \to x_M} \theta(x) \geq t_0 + \frac{1}{M} \lim_{x \to x_M} \log \frac{x_M - x_0}{x_M - x} = \infty.$

Lemma: Suppose $v(x_0) \neq 0$

(a) If $v(x_0) > 0$ and $x_M \in \Omega$, then $\omega_+ = \infty$

(b) If $v(x_0) < 0$ and $x_M \in \Omega$ then $\omega_- = -\infty$

(c) If $v(x_0) < 0$ and $x_m \in \Omega$ then $\omega_+ = \infty$

(d) If $v(x_0) > 0$ and $x_m \in \Omega$ then $\omega_- = -\infty$.

Proof:

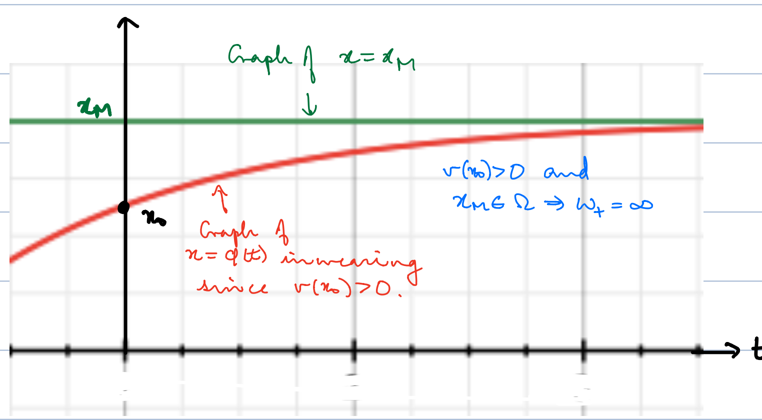
We have proved (a).

Part (b) is obtained by applying (a) to $(*)_{tr}$.

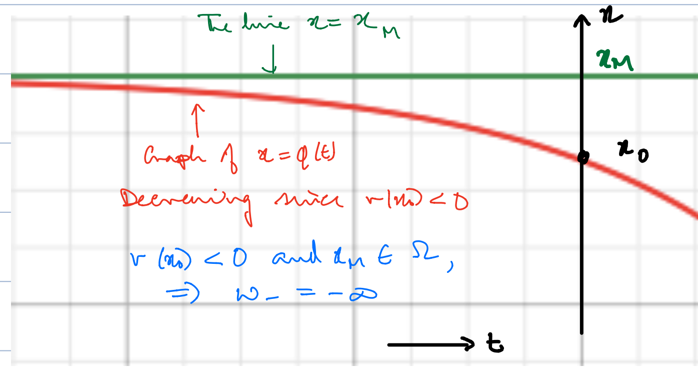
Part (c) is obtained by applying (a) to $(*)_{br}$

Part (d) is obtained by applying (b) to $(*)_{br}$, or by applying (c) to $(*)_{tr}$.

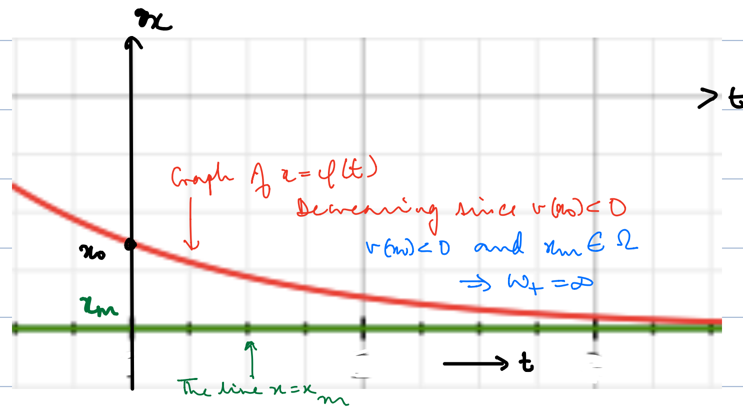
Picture for (a)



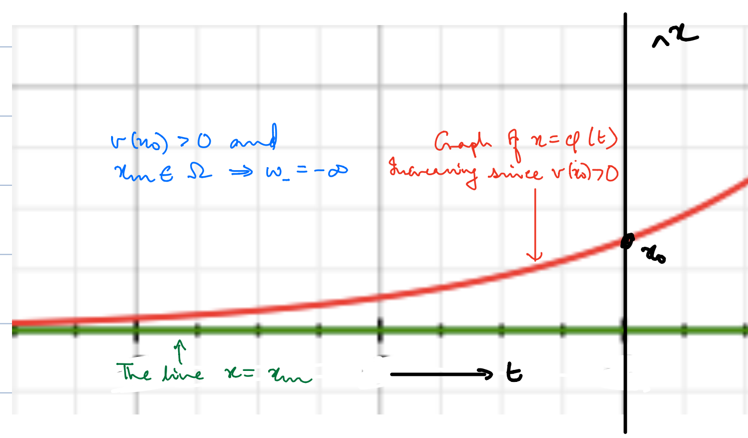
Picture for (b)



Picture for (c)



Picture for (d)



P.T.O $\longrightarrow$

Proposition: Let $\varphi : (a,b) \longrightarrow \Omega$ be a solution of $(*)$ with $x_0 \in \Omega^{reg}$. Let θ, φ_{max}, x_m, x_M, ω_-, ω_+ etc be as above. Then

(a) φ takes values in Ω^{reg}

(b) $(a,b) \subseteq (-\omega_-, \omega_+)$

(c) $\varphi = \varphi_{max}\big|_{(a,b)}$

Proof: Parts (b) and (c) follow from (a) in view of $(**)$. It remains to prove (a).

We will do this next time.