LECTURES 26 AND 27

Dates of the Lectures: April 12 and April 19, 2021

As always, $\mathbf{K} \in \{\mathbf{R},\, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\|\ \|_2$ and we will simply denote it as $\|\ \|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\|\ \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n,\, \mathbf{K}^n)$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. Lyapunov's direct method

Let U be open in $\mathbf{R}^n$ and $\boldsymbol{v}\colon U \to \mathbf{R}^n$ a $\mathscr{C}^1$ vector field on U. Let as (Δ) be the differential equation

$$(\Delta) \qquad \dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}).$$

We use the terms *equilibrium point of* (Δ) and *equilibrium point of* $\boldsymbol{v}$ interchangeably. Both mean a singular point of $\boldsymbol{v}$, i.e. a point on which $\boldsymbol{v}$ vanishes. Let $\Omega = \mathbf{R} \times U$, and for $(\tau,\, \boldsymbol{a}) \in \Omega$, let $(\Delta)_{(\tau,\, \boldsymbol{a})}$, $\boldsymbol{\varphi}_{(\tau,\, \boldsymbol{a})}$, $J(\tau,\, \boldsymbol{a})$ have their usual meaning. We make one simplification in our notation soup: when $\tau = 0$ we suppress τ. Thus $J(\boldsymbol{a}) = J(0,\, \boldsymbol{a})$, $\boldsymbol{\varphi}_{\boldsymbol{a}} = \boldsymbol{\varphi}_{(0,\, \boldsymbol{a})}$ etc. We point out that since (Δ) is autonomous, the initial time is of little importance.

Finally, we set

$$(1) \qquad g^t\boldsymbol{x} = \boldsymbol{\varphi}_{\boldsymbol{x}}(t) \qquad (\boldsymbol{x} \in U, t \in J(\boldsymbol{x})).$$

[1] See §§2.1 of Lecture 5 of ANA2.

An *equilibrium point of the flow* $\{g^t\}$ is an equilibrium point of (Δ). Technically, from the dynamical systems point of view, the notion of an equilibrium of stationary point applies to a flow $\{g^t\}$. This term when applied to (Δ) or $\boldsymbol{v}$ is a small abuse of terminology. However, (Δ) or $\boldsymbol{v}$ contain all the information of the flow $\{g^t\}$ and conversely, the flow contains all the information of (Δ) as well as of $\boldsymbol{v}$.

1.1. **Interval of existence near a stable equilibrium point.** Recall that an equilibrium position $\boldsymbol{x}_0$ of (Δ) is said to be *stable (in Lyapunov's sense)* if given $\varepsilon > 0$, there exists $\delta > 0$ such that $B(\boldsymbol{x}_0, \delta) \subset U$ and for $\boldsymbol{x} \in B(\boldsymbol{x}_0, \delta)$, the solution $\boldsymbol{\varphi}_{\boldsymbol{x}}$ of (Δ) satisfies the inequality $\|\boldsymbol{\varphi}_{\boldsymbol{x}}(t) - \boldsymbol{x}_0\| < \varepsilon$ for all $t \in [0, \infty) \cap J(\boldsymbol{x})$.

Setting $t = 0$ we see that $\delta < \varepsilon$.

Recall further that an equilibrium point $\boldsymbol{x}_0$ is said to be asymptotically stable in the sense of Lyapunov if there exists $\delta > 0$ such that $B(\boldsymbol{x}_0, \delta) \subset U$ and $\lim_{t\to\infty} g^t\boldsymbol{x} = \boldsymbol{x}_0$ for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, \delta)$. The reader may wonder how one can let t approach ∞, since $J(\boldsymbol{x})$ may not contain $[0, \infty)$. The following lemma answers that question.

Lemma 1.1.1. *Suppose $\boldsymbol{x}_0$ is a stable equilibrium point of the dynamical system represented by the differential equation (Δ) and suppose $0 < \delta < \varepsilon$ are real numbers such that $\overline{B}(\boldsymbol{x}_0, \varepsilon) \subset U$ and such that $g^t\boldsymbol{x} \in B(\boldsymbol{x}_0, \varepsilon)$ for all $t \in [0, \infty) \cap J(\boldsymbol{x})$ whenever $\boldsymbol{x} \in B(\boldsymbol{x}_0, \delta)$. Then $[0, \infty) \cap J(\boldsymbol{x}) = [0, \infty)$.*

Proof. Let $J(\boldsymbol{x}) = (\omega_-, \omega_+)$. Let T be a positive time point. Then $K_T = [0, T] \times \overline{B}(\boldsymbol{x}_0, \varepsilon)$ is a compact subset of Ω. We know then that $(t, g^t\boldsymbol{x})$ must leave K_T as $t \uparrow \omega_+$. Moreover $g^t\boldsymbol{x} \in B(\boldsymbol{x}_0, \varepsilon)$ for $t \in [0, T] \cap J(\boldsymbol{x})$. It follows that $(t, \boldsymbol{x})$ must hit $\{T\} \times B(\boldsymbol{x}_0, \varepsilon)$, whence $T \in J(\boldsymbol{x})$. Since $T > 0$ was arbitrary, we are done. □

1.2. **Lyapunov functions.** Fix an equilibrium point $\boldsymbol{x}_0$ of (Δ). Let V be an open neighbourhood of $\boldsymbol{x}_0$ in U and $Q\colon V \to \mathbf{R}$ a continuous map such that Q is $\mathscr{C}^1$ on $V \smallsetminus \{\boldsymbol{x}_0\}$. In this case, define $\dot{Q}\colon V \to \mathbf{R}$ by the formula

$$\dot{Q}(\boldsymbol{x}) = \begin{cases} \langle \boldsymbol{\nabla} Q(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{x})\rangle, & \text{if } \boldsymbol{x} \neq \boldsymbol{x}_0 \\ \mathbf{0}, & \text{if } \boldsymbol{x} = \boldsymbol{x}_0. \end{cases} \tag{1.2.1}$$

Let Q be as above. It is said to be a *Lyapunov function* for $\boldsymbol{v}$ (or (Δ), or the flow $\{g^t\}$) at $\boldsymbol{x}_0$ if in addition to the above conditions, it also satisfies

1. $Q(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in V \smallsetminus \{\boldsymbol{x}_0\}$, $Q(\boldsymbol{x}_0) = 0$.
2. $\dot{Q}(\boldsymbol{x}) \leq 0$ for $\boldsymbol{x} \in V \smallsetminus \{\boldsymbol{x}_0\}$.

If in addition we have

3. $\dot{Q}(\boldsymbol{x}) < 0$ for $\boldsymbol{x} \in V \smallsetminus \{\boldsymbol{x}_0\}$.

then Q is called a *strict Lyapunov function* at $\boldsymbol{x}_0$ for $\boldsymbol{v}$.

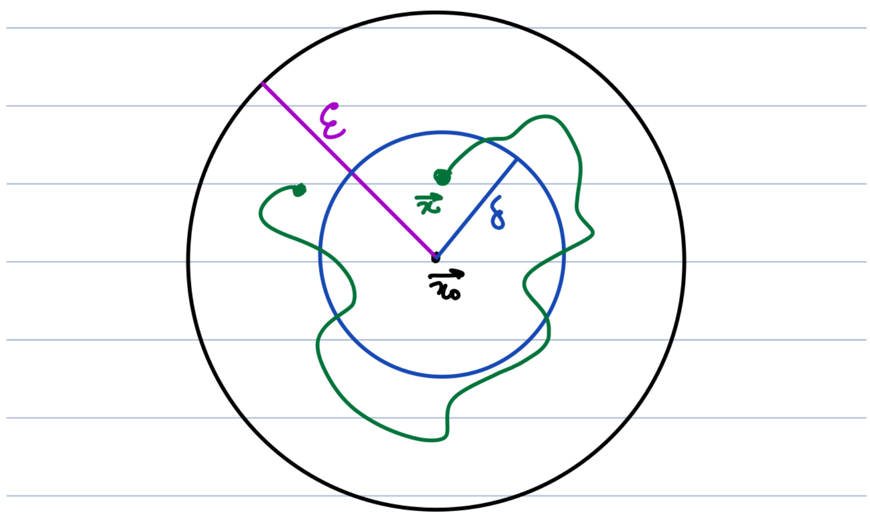
1.2.2. If Q is a strict Lyapunov function at $\boldsymbol{x}_0$ then $\boldsymbol{x}_0$ is the only equilibrium point of $\boldsymbol{v}$ in the domain V of Q. Indeed, if $\boldsymbol{x} \in V \smallsetminus \{\boldsymbol{x}_0\}$, then by definition of a strict Lyapunov function, $\langle \boldsymbol{\nabla} Q(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{x})\rangle < 0$, whence $\boldsymbol{v}(\boldsymbol{x}) \neq \mathbf{0}$.

If the vector field $\boldsymbol{v}$ and the equilibrium point $\boldsymbol{x}_0$ are understood, we often shorten the phrases "Lyapunov function at $\boldsymbol{x}_0$ for $\boldsymbol{v}$" and "strict Lyapunov function at $\boldsymbol{x}_0$ for $\boldsymbol{v}$" to "Lyapunov function" and "strict Lyapunov function" respectively.

The proof of the following theorem follows the one given in [C, Theorem 1.55, p.29].

Theorem 1.2.3. *Let $\boldsymbol{x}_0$ be an equilibrium point for (Δ). If we have a Lyapunov function Q at $\boldsymbol{x}_0$ for $\boldsymbol{v}$. Then $\boldsymbol{x}_0$ is a stable equilibrium point. If Q is a strict Lyapunov function, then $\boldsymbol{x}_0$ is an asymptotically stable equilibrium point.*

Proof. Let $Q\colon V \to \mathbf{R}$ be a Lyapunov function at $\boldsymbol{x}_0$. Suppose $\varepsilon > 0$ is given. We have to find $\delta > 0$ such that $g^t\boldsymbol{x} \in B(\boldsymbol{x}_0, \varepsilon)$ for all $t \in [0, \infty] \cap J(\boldsymbol{x})$ whenever $\boldsymbol{x} \in B(\boldsymbol{x}_0\,\delta) \cap U$. Without loss of generality, we may assume $\overline{B}(\boldsymbol{x}_0, \varepsilon) \subset V$. Let $S = S(\boldsymbol{x}_0, \varepsilon)$ be the set of points $\boldsymbol{y}$ in $\mathbf{R}^n$ such that $\|\boldsymbol{y} - \boldsymbol{x}_0\| = \varepsilon$. Then $S \subset V$. Let m be the infimum of Q on S. Since Q is continuous and $Q(\boldsymbol{x}_0) = 0$, there exists $\delta > 0$ such that $B(\boldsymbol{x}_0, \delta) \subset V$, and $Q(\boldsymbol{x}) < m/2$ for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, \delta)$. Let $\boldsymbol{x} \in B(\boldsymbol{x}_0, \delta)$. Set $J_{\geq 0} = [0, \infty) \cap J(\boldsymbol{x})$. Since $\dot{Q}(\boldsymbol{x}) \leq 0$ for $\boldsymbol{x} \in V \setminus \{\boldsymbol{x}_0\}$, we see easily that $Q(g^t\boldsymbol{x})$ is a non-increasing function of t in $J_{\geq 0}$. It follows that $Q(g^t\boldsymbol{x}) \leq m/2$ for all $t \in J_{\geq 0}$. Thus $g^t\boldsymbol{x} \notin S$ for any $t \in J_{\geq 0}$. It follows that $g^t\boldsymbol{x} \in B(\boldsymbol{x}_0, \varepsilon)$ for all $t \in J_{\geq 0}$. By definition of stability, $\boldsymbol{x}_0$ is a stable equilibrium point. In particular, $J_{\geq 0} = [0, \infty)$. It is worth pointing out, by setting $t = 0$, that $\delta < \varepsilon$.



Now suppose Q is a strict Lyapunov function. We wish to prove that there is an open ball B in U, centred at $\boldsymbol{x}_0$, such that $\lim_{t\to\infty} g^t\boldsymbol{x} = \boldsymbol{x}_0$ for all $\boldsymbol{x} \in B$. Let B_1 be an open ball in $\mathbf{R}^n$ centred at $\boldsymbol{x}_0$, with $\overline{B}_1 \subset V$. Since $\boldsymbol{x}_0$ is a stable equilibrium point, there exists an open ball B centred at $\boldsymbol{x}_0$ such that $B \subset \overline{B}_1$ and such that for every $\boldsymbol{x} \in B$, $[0, \infty) \subset J(\boldsymbol{x})$ and $g^t\boldsymbol{x} \in B_1$ for $t \in [0, \in \infty)$. In fact, by the proof above, if S_1 is the sphere centred at $\boldsymbol{x}_0$ which is the boundary of B_1, then we pick B such that for $]bdy \in B$, $Q(\boldsymbol{y})$ is less than half the infimum of Q on S_1. Let $\boldsymbol{x} \in B$. If $\boldsymbol{x} = \boldsymbol{x}_0$, then clearly $\lim_{t\to\infty} g^t\boldsymbol{x} = \boldsymbol{x}_0$. So assume $\boldsymbol{x} \neq \boldsymbol{x}_0$. Since $\dot{Q}$ is negative on $B \setminus \{\boldsymbol{x}_0\}$, we have $Q(g^t\boldsymbol{x})$ is a decreasing function of $t \geq 0$ as $t \uparrow \infty$. Let

$$\beta := \inf_{t\in[0,\,\infty)} Q(g^t\boldsymbol{x}).$$

By definition of infimum, and since $Q(g^t\boldsymbol{x})$ is decreasing, there is a sequence of time points

$$0 = t_0 < t_1 < t_2 < \cdots < t_k < \ldots \uparrow \infty$$

such that

$$\lim_{t\to\infty} Q(g^{t_k}\boldsymbol{x}) = \beta.$$

Since $\overline{B}_1$ is compact, there is a subsequence of $\{g^{t_k}\boldsymbol{x}\}$ which is convergent, and by replacing $\{g^{t_k}\boldsymbol{x}\}$ by this subsequence if necessary, we may assume $\{g^{t_k}\boldsymbol{x}\}$ is convergent. Let

$$\boldsymbol{x}^* = \lim_{k\to\infty} g^{t_k}\boldsymbol{x}.$$

Since $Q(\boldsymbol{x}^*) < Q(\boldsymbol{x})$ (we are using the fact that $\dot{Q}$ is negative off the point $\boldsymbol{x}_0$), it follows that $Q(\boldsymbol{x}^*) < \frac{1}{2}\inf_{\boldsymbol{y}\in S_1} Q(\boldsymbol{y})$, whence $g^t\boldsymbol{x}^*$ is defined for all $t \in [0,\, \infty)$. By the continuity of g^1, we see that

$$g^1\boldsymbol{x}^* = \lim_{k\to\infty} g^1 g^{t_k}\boldsymbol{x} = \lim_{k\to\infty} g^{t_k+1}\boldsymbol{x}.$$

Further, $Q(g^{t_k+1}\boldsymbol{x}) < Q(g^{t_k}\boldsymbol{x})$ for all $k \geq 0$, since $\dot{Q} < 0$ on $V \setminus \{\boldsymbol{x}_0\}$. One taking limits we get

$$Q(g^1\boldsymbol{x}^*) \leq Q(\boldsymbol{x}^*).$$

On the other hand, by definition of limits, given $\eta > 0$, there exists $K \geq 0$ such that $Q(g^1\boldsymbol{x}^*) < Q(g^{t_k+1}\boldsymbol{x}) < Q(g^1\boldsymbol{x}^*)+\eta$ for all $k \geq K$. Pick $l \geq 1$ such that $t_l > t_K+1$. Then we have $Q(g^1\boldsymbol{x}^*) \leq Q(\boldsymbol{x}^*) < Q(g^{t_l}\boldsymbol{x}) < Q(g^{t_K+1}\boldsymbol{x}) < Q(g^1\boldsymbol{x}^*) + \eta$, giving

$$Q(g^1\boldsymbol{x}^*) \leq Q(\boldsymbol{x}^*) < Q(g^1\boldsymbol{x}^*) + \eta$$

for every $\eta > 0$. It follows that $Q(g^1\boldsymbol{x}^*) = Q(\boldsymbol{x}^*)$. Once again, using the fact that $\dot{Q}(\boldsymbol{y}) < 0$ for all $\boldsymbol{y} \in V \setminus \{\boldsymbol{x}_0\}$, we see that this can only happen if $\boldsymbol{x}^* = \boldsymbol{x}_0$. Thus

$$\lim_{k\to\infty} g^{t_k}\boldsymbol{x} = \boldsymbol{x}_0.$$

Now suppose $\varepsilon > 0$ is given. Suppose ε is small enough that $B(\boldsymbol{x}_0,\, \varepsilon) \subset B$. Let

$$\gamma = \inf_{\boldsymbol{y}\in S(\boldsymbol{x}_0,\, \varepsilon)} Q(\boldsymbol{y}).$$

By the continuity of Q, we can find $\delta > 0$ such that $Q(\boldsymbol{y}) < \frac{1}{2}\gamma$ for all $\boldsymbol{y} \in B(\boldsymbol{x}_0,\, \delta)$. By a now familiar argument,

$$g^t\boldsymbol{y} \in B(\boldsymbol{x}_0,\, \varepsilon), \quad t \geq 0.$$

Now there exists $K \geq 0$ such that $\|g^{t_k}\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$ for $k \geq K$. Thus $g^{t_k}\boldsymbol{x} \in B(\boldsymbol{x}_0,\, \delta)$ for all $k \geq K$. Setting $\boldsymbol{y} = g^{t_K}\boldsymbol{x}$ in the above displayed relation, we see that $g^{t+t_K}\boldsymbol{x} \in B(\boldsymbol{x}_0,\, \varepsilon)$ for all $t \geq 0$. This is the same as saying

$$g^t\boldsymbol{x} \in B(\boldsymbol{x}_0,\, \varepsilon), \quad t \geq t_K.$$

It follows that $\lim_{t\to\infty} g^t\boldsymbol{x} = \boldsymbol{x}_0$. $\square$

References

[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, Third Edition, translated by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.

[B] J. Burke, Notes of course given at University of Washington, http://sites.math.washington.edu/~burke/crs/555/555_notes/continuity.pdf.

[C] C. Chicone, *Ordinary Differential Equations with Applications*, Springer, 2006.

[CL] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[G] C.P. Grant, *Theory of Ordinary Differential Equations*. https://www.math.utah.edu/~treiberg/GrantTodes2008.pdf, Brigham Young University.