## LECTURE 25

**Date of the Lecture:** April 7, 2021

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An $n$-tuple $(x_1, \dots, x_n)$ of symbols ($x_i$ not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set $S$ to a product set $T_1 \times \cdots \times T_n$ will often be written as an $n$-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with $f_i$ a map from $S$ to $T_i$, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \ \|_2$ and we will simply denote it as $\| \ \|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\| \ \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n, \mathbf{K}^n)$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

### 1. Parameters and initial conditions

1.1. **Formulation.** There are two kinds of initial value problems, those with a fixed "vector field" but varying initial conditions, and those with a varying vector field (which depends upon a parameter, but a fixed initial condition. We have studied the former. The kinds of problems are related, and each can be transformed into a problem of the other kind.

**1.1.1.** We have studied initial value problems of the form

$$(\Delta)_{(\tau,\boldsymbol{a})} \qquad \dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x}), \quad \boldsymbol{x}(\tau) = \boldsymbol{a}$$

where $\boldsymbol{v}\colon \Omega \to \mathbf{R}^n$ is $\mathscr{C}^1$, with $\Omega$ an open subset of $\mathbf{R} \times \mathbf{R}^n = \mathbf{R}^{n+1}$. In this case have seen that the solutions of $(\Delta)_{\boldsymbol{\xi}}$, with $\boldsymbol{\xi} = (\tau, \boldsymbol{a})$, vary in a $\mathscr{C}^1$ manner with the inputs $t$, $\tau$, and $\boldsymbol{a}$. As always, if $\boldsymbol{\xi} \in \Omega$, then $J(\boldsymbol{\xi})$ denotes the maximal interval of existence for the solution $\boldsymbol{\varphi}_{\boldsymbol{\xi}}$ of $(\Delta)_{\boldsymbol{\xi}}$. As before, let $\widetilde{\Omega} = \{(t, \tau, \boldsymbol{a}) \mid t \in J(\tau, \boldsymbol{a})\}$..

[1] See §§2.1 of Lecture 5 of ANA2.

Then $(t, \tau, \boldsymbol{a}) \mapsto \boldsymbol{\varphi}_{(\tau,\,\boldsymbol{a})}(t)$ is $\mathscr{C}^1$, where $(t, \tau, \boldsymbol{a})$ varies in $\widetilde{\Omega}$. In this formulation, $\boldsymbol{v}$ is fixed but the initial conditions $\boldsymbol{x}(\tau) = \boldsymbol{a}$ vary.

**1.1.2.** There is another question we can ask. Suppose the map $\boldsymbol{v}$ varies with a parameter, and the parameter varies (in some nice way) over a parameter space $\Pi$. Suppose the initial conditions are fixed (unlike the case just discussed). How well do the solutions vary with the parameter? As it turns out, the two questions are at the bottom the same as we will show. A precise formulation of the problem of IVP's varying with a parameter is this. Let $\Pi$ (the parameter space) be an open subset of $\mathbf{R}^d$. Suppose we have a family of $\mathscr{C}^1$ maps $\boldsymbol{w}_{\boldsymbol{\mu}} \colon \Sigma_{\boldsymbol{\mu}} \to \mathbf{R}^m$, one for each $\boldsymbol{\mu} \in \Pi$, where $\Sigma_{\boldsymbol{\mu}}$ is an open subset of $\mathbf{R} \times \mathbf{R}^m$ for each $\boldsymbol{\mu} \in \Pi$. Suppose we we have a fixed point $(\theta_0,\, \boldsymbol{b_0})$ in $\mathbf{R} \times \mathbf{R}^m$ such that $(\theta_0,\, \boldsymbol{b_0}) \in \Sigma_{\boldsymbol{\mu}}$ for every $\boldsymbol{\mu} \in \Pi$. Then we have a family of associated IVPs, one for each $\boldsymbol{\mu} \in \Pi$, namely:

$$(D)^{\boldsymbol{\mu}} \qquad \dot{\boldsymbol{z}} = \boldsymbol{w}_{\boldsymbol{\mu}}(t,\, \boldsymbol{z}), \quad \boldsymbol{z}(\theta_0) = \boldsymbol{b_0}$$

Note that the initial conditions remain fixed for all $(D)^{\boldsymbol{\mu}}$. What varies is $\boldsymbol{\mu}$. Let $\boldsymbol{\psi}_{\boldsymbol{\mu}}$ be the solution of $(D)^{\boldsymbol{\mu}}$ and $\mathcal{J}(\boldsymbol{\mu})$ the maximal interval of existence of $\boldsymbol{\psi}_{\boldsymbol{\mu}}$. Note that $\theta_0 \in \cap_{\boldsymbol{\mu}\in\Pi}\mathcal{J}(\boldsymbol{\mu})$. We would like to understand how well $(t, \boldsymbol{\mu}) \mapsto \boldsymbol{\psi}_{\boldsymbol{\mu}}(t)$ behaves as $(t,\, \boldsymbol{\mu})$ vary in the set $\widetilde{\widetilde{\Pi}} = \{(t,\, \boldsymbol{\mu}) \in \mathbf{R} \times \Pi \mid t \in \mathcal{J}(\boldsymbol{\mu})\}$. We will of course need to impose conditions on $\{\boldsymbol{w}_{\boldsymbol{\mu}}\}_{\boldsymbol{\mu}}$ to get good results. We impose the following very natural conditions

- **(a)** $(\theta_0,\, \boldsymbol{b_0}) \in \Sigma_{\boldsymbol{\mu}}$ for every $\boldsymbol{\mu} \in \Pi$.
- **(b)** The set $\Sigma = \cup_{\boldsymbol{\mu}\in\Pi}(\Sigma_{\boldsymbol{\mu}} \times \{\boldsymbol{\mu}\})$ is an open subset $\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^d = \mathbf{R}^{m+d+1}$.
- **(c)** The map $(t, \boldsymbol{x}, \boldsymbol{\mu}) \mapsto \boldsymbol{w}_{\boldsymbol{\mu}}(t,\, \boldsymbol{x})$ on $\Sigma$ is $\mathscr{C}^1$. In other words the map

$$\begin{aligned} \boldsymbol{w} \colon \Sigma &\longrightarrow \mathbf{R}^m \\ (t,\, \boldsymbol{z},\, \boldsymbol{\mu}) &\longmapsto \boldsymbol{w}_{\boldsymbol{\mu}}(t,\, \boldsymbol{z}) \end{aligned}$$

  is $\mathscr{C}^1$.

The conclusion we wish to draw is that the set $\widetilde{\widetilde{\Pi}}$ we described above is open in $\mathbf{R}^{d+1}$ and $\psi_{\boldsymbol{\mu}}(t)$ varies smoothly with $(t,\, \boldsymbol{\mu})$ as $(t,\, \boldsymbol{\mu})$ varies in $\widetilde{\widetilde{\Pi}}$.

1.2. **Transforming initial conditions into parameters.** Consider the situation in §§ 1.1.1 above. We use the notations there. We want to convert the problem $(\Delta)_{\boldsymbol{\xi}}$ to one of the form $(D)^{\boldsymbol{\mu}}$ in §§ 1.1.2. Let $\Pi = \Omega$ so that $d = n + 1$. We will define $\Sigma_{\boldsymbol{\mu}}$ below as an open subset of $\mathbf{R} \times \mathbf{R}^n = \mathbf{R}^{n+1}$ for each $\boldsymbol{\mu} \in \Pi = \Omega$. Note that this forces $m$ to equal $n$ and $\Sigma$ (which is yet to be defined) will therefore be an open subset of $\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^d = \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$. We will define $\Sigma$ first.

Since addition is a continuous map from $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ to $\mathbf{R}^{n+1}$, the inverse image of $\Omega$ under this map is an open set $U$ in $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$. On the other hand the second projection $\pi_2 \colon \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$, i.e. the map $(\boldsymbol{\xi}, \boldsymbol{\mu}) \mapsto \boldsymbol{\mu}$, is continuous, whence $\Sigma = U \cap \pi_2^{-1}(\Pi)$ is open in $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$. It is clear that the points $(\mathbf{0},\, \boldsymbol{\mu})$ lie in $\Sigma$ as $\boldsymbol{\mu}$ varies in $\Pi$. For $\boldsymbol{\mu} \in \Pi$ set

$$\Sigma_{\boldsymbol{\mu}} = \{\boldsymbol{\xi} \in \mathbf{R}^{n+1} \mid (\boldsymbol{\xi},\, \boldsymbol{\mu}) \in U\}.$$

It is clear that $\Sigma_{\boldsymbol{\mu}}$ is open in $\mathbf{R}^{n+1}$. Indeed, identifying $\mathbf{R}^{n+1}$ with $\pi_2^{-1}(\boldsymbol{\mu})$ in the obvious way, then $\Sigma_{\boldsymbol{\mu}}$ gets identified with $U \cap \pi_2^{-1}(\boldsymbol{\mu}) = \Sigma \cap \pi_2^{-1}(\boldsymbol{\mu})$.

For $\boldsymbol{\mu} \in \Pi$ let $\boldsymbol{w}_{\boldsymbol{\mu}} \colon \Sigma_{\boldsymbol{\mu}} \to \mathbf{R}^n$ be the map

$$\boldsymbol{w}_{\boldsymbol{\mu}}(\boldsymbol{\xi}) = \boldsymbol{v}(\boldsymbol{\xi} + \boldsymbol{\mu}). \tag{1.2.1}$$

Clearly $\boldsymbol{w}_{\boldsymbol{\mu}}$ is $\mathscr{C}^1$. It is easy to check that properties **(a)**, **(b)**, and **(c)** of §§1.1.2 are satisfied by our data $(\Sigma, \Pi, \{\boldsymbol{w}\}_{\boldsymbol{\mu}})$. For example, the map $(\boldsymbol{\xi},\, \boldsymbol{\mu}) \mapsto \boldsymbol{w}_{\boldsymbol{\mu}}(\boldsymbol{\xi})$ is $\mathscr{C}^1$ on $\Sigma$, since $\boldsymbol{v}$ is $\mathscr{C}^1$ on $\Omega$ and addition is a $\mathscr{C}^\infty$ operation on $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$. We are thus in the situation discussed in §§1.1.2. We have the family of IVPs $\{(D)^{\boldsymbol{\mu}}\}_{\boldsymbol{\mu}}$ associated with the family of $\mathscr{C}^1$ maps $\{\boldsymbol{w}_{\boldsymbol{\mu}}\}_{\boldsymbol{\mu}}$. As in §§1.1.1 and §§1.1.2, for $\boldsymbol{\mu} \in \Pi = \Omega$, we write $\boldsymbol{\varphi}_{\boldsymbol{\mu}}$ for the solution of $(\Delta)_{\boldsymbol{\mu}}$ and $\boldsymbol{\psi}_{\boldsymbol{\mu}}$ for the solution of $(D)^{\boldsymbol{\mu}}$. Further, $J(\boldsymbol{\mu})$ and $\mathscr{J}(\boldsymbol{\mu})$ denote their respective maximal intervals of existence.

For $\boldsymbol{\mu} \in \Pi$ let $\tau_{\boldsymbol{\mu}} \in \mathbf{R}$ and $\boldsymbol{a}_{\boldsymbol{\mu}} \in \mathbf{R}^n$ be the unique elements such that $\boldsymbol{\mu} = (\tau_\mu,\, \boldsymbol{a}_{\boldsymbol{\mu}})$ in the decomposition $\mathbf{R}^{n+1} = \mathbf{R}^{\times}\mathbf{R}^n$. It is straightforward to check that for $\boldsymbol{\mu} \in \Pi$ we have

$$\boldsymbol{\psi}_{\boldsymbol{\mu}}(t) = \boldsymbol{\varphi}_{\boldsymbol{\mu}}(t + \tau_{\boldsymbol{\mu}}) - \boldsymbol{a}_{\boldsymbol{\mu}}, \tag{1.2.2}$$

i.e.

$$\boldsymbol{\varphi}_{\boldsymbol{\mu}}(t) = \boldsymbol{\psi}(t - \tau_{\boldsymbol{\mu}}) + \boldsymbol{a}_{\boldsymbol{\mu}}. \tag{1.2.3}$$

These relations have to be interpreted as follows. Whenever one side of the equation is sensible, so is the other side. It follows that

$$J(\boldsymbol{\mu}) = \mathscr{J}(\boldsymbol{\mu}) + \tau_{\boldsymbol{\mu}}, \qquad (\boldsymbol{\mu} \in \Pi). \tag{1.2.4}$$

We have to show

(i) The set $\widetilde{\Pi}$ consisting of points $(t,\, \boldsymbol{\mu})$ in $\mathbf{R} \times \mathbf{R}^d = \mathbf{R} \times \mathbf{R}^n + 1$ such that $\boldsymbol{\mu} \in \Pi$ and $t \in \mathscr{J}(\boldsymbol{\mu})$, is open in $\mathbf{R} \times \mathbf{R}^{n+1}$; and
(ii) The map $(t,\, \boldsymbol{\mu}) \mapsto \boldsymbol{\psi}_{\boldsymbol{\mu}}(t)$ on $\widetilde{\Pi}$ is $\mathscr{C}^1$.

From a previous lecture we know that $\widetilde{\Omega} = \{(t,\, \boldsymbol{\mu}) \mid \boldsymbol{\mu} \in \Omega,\, t \in J(\boldsymbol{\mu})\}$ is open in $\mathbf{R}^{n+2}$. The map $T\colon \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n$ given by $T(t,\, \tau,\, \boldsymbol{a}) = (t - \tau, \tau, \boldsymbol{a})$ is an invertible linear transformation in $\mathbf{R} \times \mathbf{R} \times \mathbf{R}^n = \mathbf{R}^{n+2}$ and hence is a homepmorphism from $\mathbf{R}^{n+2}$ to itself. Moreover, from (1.2.2), (1.2.3), and (1.2.4), we have $\widetilde{\Pi} = T(\widetilde{\Omega})$. It follows that $\widetilde{\Pi}$ is open in $\mathbf{R}^{n+2}$. From (1.2.2), it is clear that $(t,\, \boldsymbol{\mu}) \mapsto \boldsymbol{\psi}_{\boldsymbol{\mu}}(t)$ is $\mathscr{C}^1$ on $\widetilde{\Pi}$.

In summary, given an equations of the form $(\Delta)_{(\tau,\, \boldsymbol{a})}$, with $(\tau,\, \boldsymbol{a})$ varying in an open set $\Omega$ of $\mathbf{R}^{n+1}$, and $\boldsymbol{v}\colon \Omega \to \mathbf{R}^n$ a $\mathscr{C}^1$ map, we can use the transformation (1.2.1) to convert the family $\{(\Delta)\}_{\boldsymbol{\mu}}$ to a family of IVP's of the form in $\{(D)^{\boldsymbol{\mu}}\}_{\boldsymbol{\mu}}$, satisfying (a), (b), and (c) of §§ 1.1.2, and on the open set $\widetilde{\Pi}$ of $\mathbf{R}^{n+2}$, the map $(t,\, \boldsymbol{\mu}) \mapsto \boldsymbol{\psi}_{\boldsymbol{\mu}}(t)$ is $\mathscr{C}^1$.

1.3. **Transforming parameters into initial conditions.** Now consider the family of $\mathscr{C}^1$ maps $\boldsymbol{w}_{\boldsymbol{\mu}}\colon \Sigma_{\boldsymbol{\mu}} \to \mathbf{R}^m$, $\boldsymbol{\mu} \in \Pi$, considered in §§ 1.1.2. Recall, $\Pi$ is an open subset of $\mathbf{R}^d$ (the parameter space), and $\Sigma$ is an open subset of $\mathbf{R}^{m+d+1}$. We assume conditions **(a)**, **(b)**, and **(c)** in §§ 1.1.2 are satisfied. Recall that one of our assumptions is that the map $\boldsymbol{w}\colon \Sigma \to \mathbf{R}^m$ given by

$$\boldsymbol{w}(t,\, \boldsymbol{z},\, \boldsymbol{\mu}) = \boldsymbol{w}_{\boldsymbol{\mu}}(t,\, \boldsymbol{z}) \quad ((t,\, \boldsymbol{z},\, \boldsymbol{\mu}) \in \Sigma)$$

is $\mathscr{C}^1$ (see **(c)** of §§ 1.1.2). Let $\boldsymbol{v}\colon \Sigma \to \mathbf{R}^{m+d}$ be the map

$$\boldsymbol{v}(t, \boldsymbol{x}) = \begin{bmatrix} \boldsymbol{w}(t,\, \boldsymbol{x}) \\ \mathbf{0} \end{bmatrix} \qquad ((t,\, \boldsymbol{x}) \in \Sigma). \tag{1.3.1}$$

Clearly $\boldsymbol{v}$ is $\mathscr{C}^1$. For $\boldsymbol{\mu} \in \Pi$ consider the IVP

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(t,\, \boldsymbol{x}), \qquad \boldsymbol{x}(\theta_0) = \begin{bmatrix} \boldsymbol{b_0} \\ \boldsymbol{\mu} \end{bmatrix}. \tag{1.3.2}$$

It is immediate that the solution $\boldsymbol{\varphi}_{(\theta_0,\,\boldsymbol{b_0},\,\boldsymbol{\mu})}$ of (1.3.2) is $(\boldsymbol{\psi}_{\boldsymbol{\mu}},\,\boldsymbol{\mu})$. It follows that the maximal interval of existence $J(\theta_0,\,\boldsymbol{b_0},\,\boldsymbol{\mu})$ of $\boldsymbol{\varphi}_{(\theta_0,\,\boldsymbol{b_0},\,\boldsymbol{\mu})}$ is equal to the maximal interval of existence $\mathcal{J}(\boldsymbol{\mu})$ of $\boldsymbol{\psi}_{\boldsymbol{\mu}}$.

In view of these observations, using [Lecture 23, Proposition 2.1.4] and [Lecture 23, Theorem 2.2.4], we see that $\widetilde{\Pi} = \{(t,\,\boldsymbol{\mu}) \in \mathbf{R}\times\Pi \mid t \in \mathcal{J}(\boldsymbol{\mu})\}$ is open in $\mathbf{R}\times\mathbf{R}^d$ and that $(t,\,\boldsymbol{\mu}) \mapsto \boldsymbol{\psi}_{\boldsymbol{\mu}}(t)$ is a $\mathscr{C}^1$ map on $\widetilde{\Pi}$ taking values in $\mathbf{R}^m$. In other words the solutions of $(D)^{\boldsymbol{\mu}}$ behave in a $\mathscr{C}^1$ manner with respect to $\boldsymbol{\mu}$ if $\boldsymbol{w}_{\boldsymbol{\mu}}$ varies in a $\mathscr{C}^1$ manner with $\boldsymbol{\mu}$. Details can be worked out easily by the reader, but in the event the notations are confusing, see §§ 1.3.3 below.

**1.3.3.** The many decompositions of $\mathbf{R}^{m+d+2}$ and $\mathbf{R}^{m+d+1}$ we use can create confusion. Suppose $(\tau,\,\boldsymbol{b},\,\boldsymbol{\mu}) \in \Sigma \subset \mathbf{R}\times\mathbf{R}^m\times\mathbf{R}^d$. Consider the initial value problem

$$\dot{x} = \boldsymbol{v}(t,\boldsymbol{x}), \quad \boldsymbol{x}(\tau) = (\boldsymbol{b},\,\boldsymbol{\mu})$$

where $\boldsymbol{v}$ is as in (1.3.1). Note that by condition **(b)** in §§1.1.2, $\boldsymbol{\mu} \in \Pi$. Let us write $\boldsymbol{\varphi}_{(\tau,\,\boldsymbol{b},\,\boldsymbol{\mu})}$ for the solution of this IVP, and $J(\tau,\,\boldsymbol{b},\,\boldsymbol{\mu})$ for the associated maximal interval of existence. If $\widetilde{\Sigma}$ is the set of points $(t,\tau,\boldsymbol{b},\boldsymbol{\mu})$ in $\mathbf{R}\times\Sigma$ such that $t \in J(\tau,\boldsymbol{b},\boldsymbol{\mu})$, then according to [Lecture 23, Proposition 2.1.4], $\widetilde{\Sigma}$ is open in $\mathbf{R}\times\mathbf{R}\times\mathbf{R}^m\times\mathbf{R}^d = \mathbf{R}^{m+d+2}$. If $p_{23}\colon \mathbf{R}\times\mathbf{R}\times\mathbf{R}^m\times\mathbf{R}^d \to \mathbf{R}\times\mathbf{R}^m$ is the projection to the second and third factors, i.e. $p_{23}(t,\tau,\boldsymbol{b},\boldsymbol{\mu}) = (\tau,\boldsymbol{b})$, then $E_{(\theta_0,\,\boldsymbol{b_0})} = p_{23}^{-1}(\theta_0,\,\boldsymbol{b_0})$ can be identified with $\mathbf{R}\times\mathbf{R}^d$ (the topology and euclidean structures coinciding, with the topology of $E_{(\theta_0,\,\boldsymbol{b_0})}$ being the subspace topology from $\mathbf{R}^{m+d+2}$). Under this identification, and using our earlier observation that $J(\theta_0,\,\boldsymbol{b_0},\,\boldsymbol{\mu}) = \mathcal{J}(\boldsymbol{\mu})$, $\widetilde{\Pi}$ gets identified with $\widetilde{\Sigma}\cap E_{(\theta_0,\,\boldsymbol{b_0})}$. It follows that $\widetilde{\Pi}$ is open in $\mathbf{R}\times\mathbf{R}^d$. Moreover, by [Lecture 23, Theorem 2.2.4], the map $(t,\,\boldsymbol{\xi}) \mapsto \boldsymbol{\varphi}_{(\tau,\,\boldsymbol{b},\,\boldsymbol{\mu})}(t)$ is $\mathscr{C}^1$ on $\widetilde{\Sigma}$, whence its restriction to $\widetilde{\Sigma}\cap E_{(\theta_0,\,\boldsymbol{b_0})}$ is $\mathscr{C}^1$, i.e. the map $\boldsymbol{\Phi}\colon \widetilde{\Pi} \to \mathbf{R}^{m+d}$ given by $\boldsymbol{\Phi}(t,\,\boldsymbol{\mu}) = \boldsymbol{\varphi}_{(\theta_0,\,\boldsymbol{b_0},\,\boldsymbol{\mu})}(t)$ is $\mathscr{C}^1$.

In view of the discussion above we have the following theorem.

**Theorem 1.3.4.** *The map $\boldsymbol{\Psi}\colon \widetilde{\Pi} \to \mathbf{R}^m$ given by $\boldsymbol{\Psi}(t,\,\boldsymbol{\mu}) = \boldsymbol{\psi}_{\boldsymbol{\mu}}(t)$, is $\mathscr{C}^1$.*

*Proof.* Use the formula $\boldsymbol{\Phi}(t,\,\boldsymbol{\mu}) = (\boldsymbol{\Psi}(t,\,\boldsymbol{\mu}),\,\boldsymbol{\mu})$ for $(t,\,\boldsymbol{\mu}) \in \widetilde{\Pi}$, and the fact that $\boldsymbol{\Phi}$ is $\mathscr{C}^1$. □

1.4. **Multiple roots of the characteristic polynomial of a homogeneous linear DE with constant coefficients.** Consider the homogenous linear differential equation with constant coefficients

$$p(D)y = 0$$

where $p(s) = s^n + \mu_1 s^{n-1} + \cdots + \mu_{n-1}s + \mu_n$ is a polynomial of degree $n$ with real coefficients $a_1,\ldots,a_n$. We have assumed, without loss of generality, that $p$ is monic. Now suppose $\lambda_1$ and $\lambda_2$ are distinct roots of the characteristic equation $p(r) = 0$. We know that $(e^{\lambda_1 t} - e^{\lambda_2 t})/(\lambda_1 - \lambda_2)$ is a solution of (1.4.1). In fact it is the solution of the IVP

$$p(D)y = 0, \quad y(0) = 0, y'(0) = 1. \tag{1.4.1}$$

Now regard (1.4.1) as a family of IVPs indexed by $\boldsymbol{\mu} = (\mu_1,\,\ldots,\,\mu_n)$. Since the components of $\boldsymbol{\mu}$ are symmetric polynomials in the roots of $p$, we may (loosely) talk about moving $\boldsymbol{\mu}$ so that $\lambda_2 \to \lambda_1$. We know that solutions vary well with parameters, and therefore as $\lambda_2 \to \lambda_1 = \lambda$, we see that the solution $(e^{\lambda_1 t} - e^{\lambda_2 t})/(\lambda_1 - \lambda_2)$ approaches a solution of the limiting IVP, whose characteristic polynomial has a double root at $\lambda$. Thus $te^{\lambda t}$ is a solution when $\lambda_1 = \lambda_2 = \lambda$. One extend that reasoning for multiple roots of higher order of $p$. This is the reasoning Euler and Lagrange used to explain the appearance of the solutions $t^k e^{\lambda t}$, $0 \le k < \nu$, where $\nu$ is the multiplicity of the root $\lambda$ of $p$ (see [A1, **26.4**, pp.178–179]).

## 2. Local families of transformations

2.1. **Local phase flows.** (See [A1, p. 51, §§ 7.5].) Let $\boldsymbol{v}$ be a smooth vector field in the phase space $U$, and let $\boldsymbol{x_0}$ be a point in $U$. The following definition should have been given earlier in the course.

**Definition 2.1.1.** By a *local phase flow* determined by the vector field $\boldsymbol{v}$ in a neighbourhood of the point $\boldsymbol{x_0}$ we mean a triple $(I, V_0, g)$, consisting of an interval $I = \{t \in \mathbf{R} \mid \ |t| < \varepsilon\}$, and a mapping $g\colon I \times V_0 \to U$, which satisfies the following three conditions:

1) For fixed $t \in I$ the mapping $g^t\colon V_0 \to U$ defined by $g^t\boldsymbol{x} = g(t, \boldsymbol{x})$ is a diffeomorphism onto its image;
2) For fixed $\boldsymbol{x} \in V_0$ the mapping $\boldsymbol{\varphi} = \boldsymbol{\varphi_x}\colon I \to U$ defined by $\boldsymbol{\varphi}(t) = g^t\boldsymbol{x}$ is a solution of $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$ satisfying the initial condition $\boldsymbol{\varphi}(0) = \boldsymbol{x}$;
3) The group property $g^{s+t}\boldsymbol{x} = g^s(g^t\boldsymbol{x})$ holds for all $\boldsymbol{x}$, $s$ and $t$ such that the right-hand side is defined, where for every point $\boldsymbol{y} \in V_0$ there exists an open set $V$ with $\boldsymbol{y} \in V \subset V_0$ and a number $\delta > 0$ such that the right hand side is defined for $|s| < \delta$, $|t| < \delta$ and all $\boldsymbol{x} \in V$.

**Proposition 2.1.2.** *The vector field $\boldsymbol{v}$ determines a local phase flow in the neighbourhood of every phase point $\boldsymbol{x_0}$.*

*Proof.* This follows from [Lecture 20, 2.1.6] and [Lecture 23, Theorem 2.2.4]. □

## 3. Lyapunov stability and asymptotic stability

3.1. **Lyapunov stability.** Let $U$ be an open subset of $\mathbf{R}^n$ and consider the associated autonomous differential equation

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}). \tag{3.1.1}$$

where $\boldsymbol{v}$ is a vector field on $U$ of class $\mathscr{C}^2$. As before, the solution of (3.1.1) such that its value at $t = 0$ is $\boldsymbol{a}$ is denoted $\boldsymbol{\varphi_a}$ and the maximal interval of existence of $\boldsymbol{\varphi_a}$ is denoted $J(\boldsymbol{a})$.

By an *equilibrium position* of (3.1.1) we mean a point $\boldsymbol{a} \in U$ such that $\boldsymbol{v}(\boldsymbol{a}) = \mathbf{0}$. Such points are also called *stationary points* of (3.1.1) or *singular points* of $\boldsymbol{v}$. Note that if $\boldsymbol{a}$ is an equilibrium point then $\boldsymbol{\varphi_a}(t) \equiv \boldsymbol{a}$ and $J(\boldsymbol{a}) = \mathbf{R}$.

**Definition 3.1.2.** An equilibrium position $\boldsymbol{a_0}$ of (3.1.1) is said to be *stable (in Lyapunov's sense)* if given $\varepsilon > 0$, there exists $\delta > 0$ such that $B(\boldsymbol{a_0}, \delta) \subset U$ and for $\boldsymbol{a} \in B(\boldsymbol{a_0}, \delta)$, the solution $\boldsymbol{\varphi_a}$ of (3.1.1) satisfies the inequality $\|\boldsymbol{\varphi_a}(t) - \boldsymbol{a_0}\| < \varepsilon$ for all $t \in [0, \infty) \cap J(\boldsymbol{a})$ (Figure 1).

The following picture is taken from [A1]. Without loss of generality, $\boldsymbol{a_0}$ is taken to be $\mathbf{0}$.

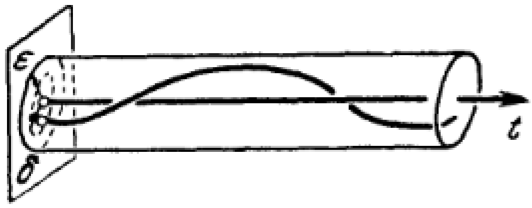


Figure 1. This is Fig. 157 on p. 155 of [A1].

### 3.2. Asympototic stability.

**Definition 3.2.1.** An equilibrium position $\boldsymbol{a_0}$ of (3.1.1) is said to be *asymptotically stable* if it is stable in Lyapunov's sense and if there exists $\delta > 0$ such that $B(\boldsymbol{a_0}, \delta) \subset U$ and

$$\lim_{t\to\infty} \boldsymbol{\varphi_a}(t) = \boldsymbol{a_0}$$

whenever $\boldsymbol{a} \in B(\boldsymbol{a_0}, \delta)$ (Figure 2).

The following picture is taken from [A1]. Without loss of generality, $\boldsymbol{a_0}$ is taken to be $\mathbf{0}$.

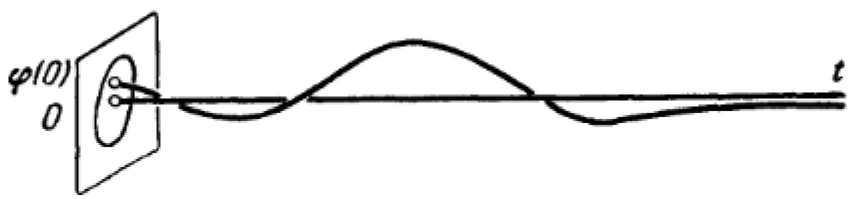


Figure 2. This is Fig. 159 on p. 156 of [A1].

**Example 3.2.2.** Consider the DE (with phase space $\mathbf{R}^2$):

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

There is only one equilibrium position, namely $\boldsymbol{a_0} = \mathbf{0}$, since the matrix $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ is non-singular. The orbits of the 1-parameter group $\{e^{tA}\}$ are circles. In fact the orbit of $(c, d) \in \mathbf{R}^2$ is the circle of radius $\sqrt{c^2 + d^2}$ centred at $\mathbf{0}$. It follows (by taking $\delta = \varepsilon/2$ for every $\varepsilon > 0$) that $\mathbf{0}$ is a Lyapunov stable equilibrium point. On the other hand, because the phase curves are circles, it is clear that $\mathbf{0}$ cannot be an asymptotically stable equilibrium position.

## References


[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, Third Edition, translated by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.

[B] J. Burke, Notes of course given at University of Washington, http://sites.math.washington.edu/~burke/crs/555/555_notes/continuity.pdf.

[CL] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[G] C.P. Grant, *Theory of Ordinary Differential Equations*. https://www.math.utah.edu/~treiberg/GrantTodes2008.pdf, Brigham Young University.