LECTURE 24

Dates of the Lectures: April 5, 2021

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \, \|_2$ and we will simply denote it as $\| \, \|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\| \, \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n, \mathbf{K}^n)$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. Rectification

Throughout this lecture U is an open set of $\mathbf{R}^n$, $\boldsymbol{v} \colon U \to \mathbf{R}^n$ a $\mathscr{C}^1$ vector field.

1.1. **The rectification theorem.** Let $\boldsymbol{a_0} \in U$ be a *regular point* of $\boldsymbol{v}$, i.e. $\boldsymbol{v}(\boldsymbol{a_0}) \neq 0$. Our goal is to find an open neighbourhood V in U of $\boldsymbol{a_0}$, an open neighbourhood W of $\mathbf{0} \in \mathbf{R}^n$, and a diffeomorphism $\boldsymbol{F} \colon V \to W$ such that $\boldsymbol{F}(\boldsymbol{a_0}) = \mathbf{0}$ and $\boldsymbol{F}_*(\boldsymbol{v}) = \mathbf{e}_1$. Recall from (3.1.1) and 3.1.3 of Lectures 17 and 18 that if we have a diffeomorphism $\boldsymbol{F} \colon V \to W$ where V is open in U, then the *push-forward* of $\boldsymbol{v}$ to W is the vector field $\boldsymbol{F}_*\boldsymbol{v}$ on W given by

$$(\boldsymbol{F}_*\boldsymbol{v})(\boldsymbol{y}) = \boldsymbol{F}'(\boldsymbol{G}(\boldsymbol{y}))\boldsymbol{v}(\boldsymbol{G}(\boldsymbol{y})) \qquad (\boldsymbol{y} \in W), \tag{1.1.1}$$

where $\boldsymbol{G} = \boldsymbol{F}^{-1}$.

It is clear, by translating $\boldsymbol{a_0}$ to the origin if necessary, that we may assume $\boldsymbol{a_0} = \mathbf{0}$. Since $\boldsymbol{v}(\mathbf{0}) \neq \mathbf{0}$, at least one component of $\boldsymbol{v}(\mathbf{0}) = (v_1(\mathbf{0}), \dots, v_n(\mathbf{0}))$ is

[1]See §§2.1 of Lecture 5 of ANA2.

non-zero, and without loss of generality, we assume $v_1(\mathbf{0}) \neq 0$. In other words $\boldsymbol{v}(\mathbf{0})$ is not in the span of $\mathbf{e}_2, \ldots, \mathbf{e}_n$.

For $\boldsymbol{a} \in U$ let $\boldsymbol{\varphi}_{\boldsymbol{a}}$ be the solution of the IVP

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{a}.$$

Let $I = [-c, c]$ be an interval of existence of $\boldsymbol{\varphi}_{\mathbf{0}}$. By Theorem 3.1.6 of Lecture 20 we can find an open neighbourhood W' in U of $\mathbf{0}$ such that I is an interval of existence for $\boldsymbol{\varphi}_{\boldsymbol{a}}$ for all $\boldsymbol{a} \in W'$ (in *loc.cit.*, W' is a ball of radius δ around $\mathbf{0}$). Let

$$W = \{(t, y_2, \ldots, y_n) \in \mathbf{R}^n \mid t \in (-c, c) \text{ and } (0, y_2, \ldots, y_n) \in W'\}.$$

It is clear that W is open in $\mathbf{R}^n$, for the set of points $(y_2, \ldots, y_n) \in \mathbf{R}^{n-1}$ such that $(0, y_2, \ldots, y_n) \in W'$ is open in $\mathbf{R}^{n-1}$ and $(-c, c)$ is open in $\mathbf{R}$. Since the map $\boldsymbol{H} \colon I \times W' \to U$ given by $\boldsymbol{H}(t, \boldsymbol{y}) = \boldsymbol{\varphi}_{\boldsymbol{y}}(t)$ is $\mathscr{C}^1$ (by Theorem 3.2.3 of Lectures 21 and 22), we see that the map $\boldsymbol{G} \colon W \to U$ given by

$$\boldsymbol{G}(t, y_2, \ldots, y_n) = \boldsymbol{\varphi}_{(0, y_2, \ldots, y_n)}(t), \qquad (t, y_2, \ldots, y_n) \in W$$

is $\mathscr{C}^1$. Note that by the definition of $\boldsymbol{\varphi}_{(0, y_2, \ldots, y_n)}$ we have

$$\left.\frac{\partial \boldsymbol{G}}{\partial t}\right|_{(t, y_2, \ldots, y_n)} = \boldsymbol{v}(\boldsymbol{G}(t, y_2, \ldots, y_n)). \tag{1.1.2}$$

Moreover, $\boldsymbol{G}(0, y_2, \ldots, y_n) = (0, y_2, \ldots, y_n)$. This yields

$$\boldsymbol{G}(\mathbf{0}) = \mathbf{0}, \quad \text{and} \quad \frac{\partial \boldsymbol{G}}{\partial y_i}\Big|_{(0, y_2, \ldots, y_n)} = \mathbf{e}_i, \; i = 2, \ldots, n.$$

Thus

$$J\boldsymbol{G}(\mathbf{0}) = [\boldsymbol{v}(\mathbf{0}) \; \mathbf{e}_2 \; \ldots \; \mathbf{e}_n].$$

By our choice of co-ordinates on $\mathbf{R}^n$, we know that $\boldsymbol{v}(\mathbf{0})$ is not in the linear span of $\mathbf{e}_2, \ldots, \mathbf{e}_n$, whence the above formula shows that $J\boldsymbol{G}(\mathbf{0})$ is non-singular. By the inverse function theorem we see that $\boldsymbol{G}$ is a diffeomorphism on an open neighbourhood in W of $\mathbf{0}$ onto an open neighbourhood of $\mathbf{0}$ of $U \subset \mathbf{R}^n$. By shrinking W around $\mathbf{0}$ if necessary, we assume $\boldsymbol{G}$ is a diffeomorphism on W and write $V = \boldsymbol{G}(W)$. Then V is an open neighbourhood of $\mathbf{0}$ in U and we have a diffeomorphism $\boldsymbol{G} \colon W \xrightarrow{\sim} V$. Let $\boldsymbol{F} = \boldsymbol{G}^{-1}$. By exchanging the roles of $\boldsymbol{G}$ and $\boldsymbol{F}$ in formula (1.1.1) we see that

$$(\boldsymbol{G}_*\mathbf{e}_1)(\boldsymbol{x}) = \boldsymbol{G}'(\boldsymbol{F}(\boldsymbol{x}))\mathbf{e}_1(\boldsymbol{x}) = \boldsymbol{G}'(\boldsymbol{F}(\boldsymbol{x}))\mathbf{e}_1 = (J\boldsymbol{G}(\boldsymbol{F}(\boldsymbol{x})))\mathbf{e}_1.$$

The identity (1.1.2) shows that the first column of $J\boldsymbol{G}(\boldsymbol{F}(\boldsymbol{x}))$ is $\boldsymbol{v}(\boldsymbol{G}(\boldsymbol{F}(\boldsymbol{x}))) = \boldsymbol{v}(\boldsymbol{x})$. In other words, $(\boldsymbol{G}_*\mathbf{e}_1)(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x})$. It follows that

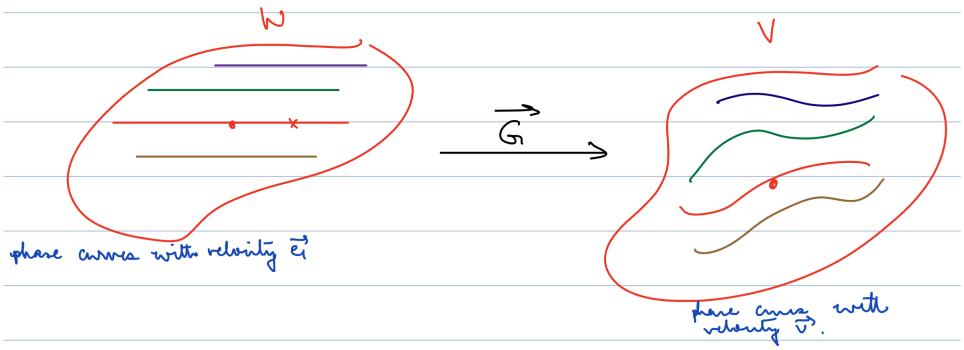
$$\boldsymbol{F}_*\boldsymbol{v} = \mathbf{e}_1.$$

We have therefore proved:

Theorem 1.1.3. (The Rectification Theorem) *Let U be an open subset of $\mathbf{R}^n$, $\boldsymbol{v} \colon U \to \mathbf{R}^n$ a $\mathscr{C}^1$ vector field on U, and $\boldsymbol{a}_{\mathbf{0}} \in U$ a regular point for $\boldsymbol{v}$, i.e. $\boldsymbol{v}(\boldsymbol{a}_{\mathbf{0}}) \neq \mathbf{0}$. Then there are open sets V and W in $\mathbf{R}^n$ with $\boldsymbol{a}_{\mathbf{0}} \in V \subset U$, and $\mathbf{0} \in W$, and a diffeomorphism $\boldsymbol{F} \colon V \xrightarrow{\sim} W$ such that*

$$\boldsymbol{F}(\boldsymbol{a}_{\mathbf{0}}) = \mathbf{0}, \quad \textit{and} \quad \boldsymbol{F}_*\boldsymbol{v} = \mathbf{e}_1.$$

1.1.4. Here is another way of seeing $\boldsymbol{G}_*\mathbf{e}_1 = \boldsymbol{v}$. Consider the smooth path $t \mapsto \boldsymbol{\psi}^{\boldsymbol{y}}(t)$ where $\boldsymbol{\psi}^{\boldsymbol{y}}(t) = t\mathbf{e}_1 + y_2\mathbf{e}_2 + \cdots + y_n\mathbf{e}_n$ for a fixed point $\boldsymbol{y} = (0, y_2, \ldots, y_{n-1}) \in \mathbf{R}^n$. Its velocity vector at every time point is $\mathbf{e}_1$. If we restrict t to $(-c, c)$ and $\sum_{i=2}^n y_i\mathbf{e}_i$ to $W' \cap \mathrm{span}\{\mathbf{e}_2, \ldots, \mathbf{e}_n\}$, the image of our smooth path $t \mapsto \boldsymbol{\psi}^{\boldsymbol{y}}(t)$ under $\boldsymbol{G}$ is the $\mathscr{C}^1$ path $t \mapsto \boldsymbol{\varphi}_{(0,y_2,\ldots,y_n)}(t)$, whose velocity vectors are $\boldsymbol{v}(\boldsymbol{\varphi}_{(0,y_2,\ldots,y_n)}(t))$. Since we can pass paths of the type $t \mapsto \boldsymbol{\psi}^{\boldsymbol{y}}(t)$ through every point in $\mathbf{R}^n$ and hence through every point of W, it is immediate that $\boldsymbol{G}_*\mathbf{e}_1 = \boldsymbol{v}$.



References

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