LECTURE 23

Dates of the Lectures: March 31, 2021

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \;\|_2$ and we will simply denote it as $\| \;\|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\| \;\|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n, \mathbf{K}^n)$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. **Summary of results from Lectures 20, 21 and 22**

1.1. We assume throughout that Ω is a domain in $\mathbf{R} \times \mathbf{R}^n = \mathbf{R}^{n+1}$ and that $\boldsymbol{v} \colon \Omega \to \mathbf{R}^n$ is $\mathscr{C}^1$. This means $\boldsymbol{v}$ is a locally Lipschitz, and since the questions we addressed in Lectures 20, 21 and 22 (continuity, differentiability) are local properties, we assume, without loss of generality, that $\boldsymbol{v}$ is Lipschitz with Lipschitz constant L. As in *loc.cit.*, we write (Δ) for the DE $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$, and if $\boldsymbol{\xi} = (\tau, \boldsymbol{a})$ is a point in Ω, we write $\Delta_{\boldsymbol{\xi}}$ for the initial value problem $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$, $\boldsymbol{x}(\tau) = \boldsymbol{a}$. As before $\boldsymbol{\varphi}_{\boldsymbol{\xi}}$ denotes the solution of $(\Delta)_{\boldsymbol{\xi}}$ and $J(\boldsymbol{\xi})$ the maximal interval of existence of $\boldsymbol{\varphi}_{\boldsymbol{\xi}}$. Given a solution $\boldsymbol{\varphi} \colon I \to \Omega$ of Δ on a closed interval $I = [c, d]$ we can find an open subset U of $I \times \mathbf{R}^n$ such that:

(a) $\overline{U}$ is a compact subset of Ω, where $\overline{U}$ is the closure of U in $I \times \mathbf{R}^n$.

[1]See §§2.1 of Lecture 5 of ANA2.

(b) $(t, \boldsymbol{\varphi}(t)) \in U$ for all $t \in I$. (However, if $\boldsymbol{\xi}$ is a point of U which is not on the graph of $\boldsymbol{\varphi}$, then $(t, \boldsymbol{\varphi}_{\boldsymbol{\xi}}(t))$ need not lie in U for every $t \in I$. These points will however lie in an open subset of $I \times \mathbf{R}^n$ that we denoted U_{δ_1} where δ_1 is a positive real number, and $\overline{U}_{\delta_1}$ is compact in Ω.)

(c) Let $\pi\colon U \to I$ be the natural projection. There exists a positive real number δ_m such that for every $\tau \in I$ we have $\pi^{-1}(\tau) = \{\tau\} \times B(\boldsymbol{\varphi}(\tau), \delta_m)$ (see last paragraph of p.4 of Lecture 20).

(d) $I \subset \bigcap_{\boldsymbol{\xi} \in U} J(\boldsymbol{\xi})$.

We thus have a map $\boldsymbol{F}\colon I \times U \longrightarrow \mathbf{R}^n$ given by $\boldsymbol{F}(t, \tau, \boldsymbol{x}) = \boldsymbol{\varphi}_{(\tau, \boldsymbol{x})}(t)$. The main results of Lectures 20, 21, and 22 are that $\boldsymbol{F}$ is continuous and the restriction of $\boldsymbol{F}$ to any of the subsets $I \times \pi^{-1}(\tau) = I \times \{\tau\} \times B(\boldsymbol{\varphi}(\tau), \delta_m)$ is $\mathscr{C}^1$ where we regard the fibre $\pi^{-1}(\tau)$ as an open ball in $\mathbf{R}^n$ of radius δ_m and centre $\boldsymbol{\varphi}(\tau)$ (see (c) above). We rephrase the last result as

Lemma 1.1.1. *For fixed $\tau \in I$, the map $\boldsymbol{F}_\tau\colon I \times B(\boldsymbol{\varphi}(\tau), \delta_m) \longrightarrow \mathbf{R}^n$ given by the rule $(t, \boldsymbol{x}) \to \boldsymbol{F}(t, \tau, \boldsymbol{x})$, is $\mathscr{C}^1$.*

1.1.2. One essential observation from the above summary is this. Suppose we have a point $(t_0,\, \tau_0,\, \boldsymbol{a}_0)$ in $\mathbf{R} \times \Omega$ such that $t_0 \in J(\tau_0,\, \boldsymbol{a}_0)$. Then there is an interval I containing t_0 and τ_0 in its interior and open ball B in $\mathbf{R}^n$ centred at $\boldsymbol{a}_0$ such that $\{\tau_0\} \times B \subset \Omega$ and $I \subset J(\tau_0, \boldsymbol{a})$ for all $\boldsymbol{a} \in B$. This is seen by setting $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{(\tau_0,\, \boldsymbol{a}_0)}$ and picking a closed interval I in $J(\tau_0,\, \boldsymbol{a}_0)$ containing t_0 and τ_0 in its interior and using $\boldsymbol{\varphi}$ and I to build the open set U above. For B pick $B(\boldsymbol{a}_0,\, \delta_m)$. It follows that on the map $\boldsymbol{F}_{\tau_0}$, i.e. the map $(t, \boldsymbol{a}) \mapsto \boldsymbol{\varphi}_{(\tau_0, \boldsymbol{a})}(t)$, is defined on $I \times B$ and is $\mathscr{C}^1$. See Figure 1 where $I \times B$ is embedded in $\mathbf{R}^{n+2}$ via the map $(t,\, \boldsymbol{a}) \mapsto (t,\, \tau_0,\, \boldsymbol{a})$

2. Differentiability of solutions of (Δ) with respect to all parameters

We continue to use the notations recalled in the summary above.

2.1. The open set $\widetilde{\Omega}$.

Let

$$\widetilde{\Omega} = \Big\{(t,\, \tau,\, \boldsymbol{x}) \in \mathbf{R}^{n+2} \,\big|\, (\tau,\, \boldsymbol{x}) \in \Omega \text{ and } t \in J(\tau,\, \boldsymbol{x})\Big\}. \tag{2.1.1}$$

We claim that $\widetilde{\Omega}$ is an open subset of $\mathbf{R}^{n+2}$. To see this we move to the autonomous differential equation associated with (Δ). Here is a reminder of how that works. Let $\boldsymbol{w}\colon \Omega \to \mathbf{R}^{n+1}$ be the map

$$\boldsymbol{w}(s, \boldsymbol{x}) = (1, \boldsymbol{v}(s, \boldsymbol{x})). \tag{2.1.2}$$

We have an associated DE

$$\dot{\boldsymbol{z}} = \boldsymbol{w}(\boldsymbol{z}). \tag{$\widetilde{\Delta}$}$$

Note that $(\widetilde{\Delta})$ is autonomous. This is the autonomous differential equation associated with (Δ). We now proceed with showing that $\widetilde{\Omega}$ is open. For $(\tau, \boldsymbol{a}) \in \Omega$ we have the IVP

$$\dot{\boldsymbol{z}} = \boldsymbol{w}(\boldsymbol{z}), \qquad \boldsymbol{z}(0) = (\tau,\, \boldsymbol{a}). \tag{$(\widetilde{\Delta})_{(\tau,\boldsymbol{a})}$}$$

We write $\boldsymbol{\psi}_{(\tau,\, \boldsymbol{a})}$ for the solution of $(\widetilde{\Delta})_{(\tau,\, \boldsymbol{a})}$ and $\widetilde{J}(\tau,\, \boldsymbol{a})$ for the maximal interval of existence for this solution. Recall that if $\boldsymbol{\psi}_{(\tau,\, \boldsymbol{a})} = (\psi_0, \psi_1, \ldots, \psi_n)$ then (a) $\psi_0(t) = t + \tau$, and (b) writing $\boldsymbol{g} = (\psi_1,\, \ldots,\, \psi_n)$ we have $\dot{\boldsymbol{g}}(t) = \boldsymbol{v}(t + \tau,\, \boldsymbol{g}(t))$. It follows that $t \mapsto \boldsymbol{g}(t - \tau)$ is a solution of $(\Delta)_{(\tau,\, \boldsymbol{a})}$ whence $\widetilde{J}(\tau,\, \boldsymbol{a}) + \tau \subset J(\tau,\, \boldsymbol{a})$.

Conversely, one checks easily that $t \mapsto (t+\tau,\, \boldsymbol{\varphi}_{(\tau,\,\boldsymbol{a})}(t+\tau)$ is a solution of $(\widetilde{\Delta})_{(\tau,\,\boldsymbol{a})}$, whence $J(\tau,\, \boldsymbol{a}) \subset \widetilde{J}(\tau,\, \boldsymbol{a}) + \tau$. Thus

$$\boldsymbol{\psi}_{(\tau,\,\boldsymbol{a})}(t) = (t+\tau,\, \boldsymbol{\varphi}_{(\tau,\,\boldsymbol{a})}(t+\tau)) \quad \text{and} \quad J(\tau,\, \boldsymbol{a}) = \widetilde{J}(\tau,\, \boldsymbol{a}) + \tau. \tag{2.1.3}$$

Note that $0 \in \widetilde{J}(\tau,\, \boldsymbol{a})$ for every $(\tau,\, \boldsymbol{a}) \in \Omega$. In order to translate results involving $\boldsymbol{\psi}_{(\tau,\,\boldsymbol{a})}$ to results involving $\boldsymbol{\varphi}_{(\tau,\,\boldsymbol{a})}$ and vice-versa we use the linear automorphism linear automorphism $T\colon \mathbf{R}^{n+2} \longrightarrow \mathbf{R}^{n+2}$ given by

$$T(t,\, \tau,\, \boldsymbol{a}) = (t+\tau,\, \tau,\, \boldsymbol{a}).$$

Let $(t_0, \tau_0, \boldsymbol{a}_0)$ be such that $t_0 \in J(\tau_0, \boldsymbol{a}_0)$. Pick $I = [c, d]$ such that $t_0, \tau_0 \in (c, d) \subset I \subset J(\tau_0, \boldsymbol{a}_0)$. This is always possible. Let $\boldsymbol{\varphi}\colon I \to \Omega$ be the restriction of $\boldsymbol{\varphi}_{(\tau_0,\,\boldsymbol{a}_0)}$ to I. Let δ_1, δ_m, U etc. be as in Subsection 1.1, where the input for defining them is the solution $\boldsymbol{\varphi}\colon I \to \Omega$ of (Δ). Let $\widetilde{I} = I - \tau_0$. Then by (2.1.3), $\widetilde{I} \subset \widetilde{J}(\tau_0,\, \boldsymbol{a}_0)$. From the observation made in 1.1.2, with $\widetilde{I}$ playing the role of I, $\boldsymbol{\psi}_{(\tau_0,\boldsymbol{a}_0)}$ the role of $\boldsymbol{\varphi}$, and 0 the role of τ_0, there is an open ball $\widetilde{B}$ in Ω centred at $(\tau_0,\, \boldsymbol{a}_0)$ such that $\widetilde{I} \subset \widetilde{J}(\tau, \boldsymbol{a})$ for all $(\tau,\, \boldsymbol{a}) \in B$. By shrinking $\widetilde{B}$ if necessary, we assume that $\widetilde{B} \subset U$.

Let $V = T((c-\tau_0, d-\tau_0) \times \widetilde{B})$. Then V is an open neighbourhood of $(t_0,\, \tau_0\, \boldsymbol{a}_0)$ in $\mathbf{R}^{n+2}$. Moreover, by] (2.1.3) we see that $V \subset \widetilde{\Omega}$. Since $(t_0,\, \tau_0,\, \boldsymbol{a}_0)$ is an arbitrary point of $\widetilde{\Omega}$, we have proven:

Proposition 2.1.4. *$\widetilde{\Omega}$ is an open subset of $\mathbf{R}^{n+2}$.*

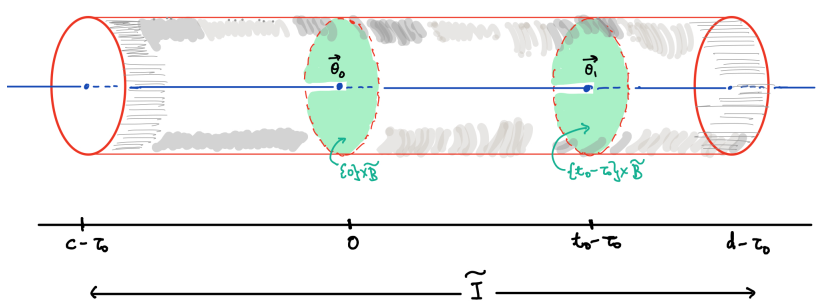

Figure 1. The cylinder $C = (c-\tau_0, d-\tau_0) \times \widetilde{B}$ with $\boldsymbol{\theta_0} = (0,\, \tau_0,\, \boldsymbol{a}_0)$ and $\boldsymbol{\theta_1} = (t_0-\tau_0,\, \tau_0,\, \boldsymbol{a}_0)$. Since C is open in $\mathbf{R}^{n+2}$, $V = T(C)$ is also open in $\mathbf{R}^{n+2}$. Moreover $(t_0,\, \tau_0,\, \boldsymbol{a}_0) \in V \subset \widetilde{\Omega}$.

2.2. **Differentiability with respect to τ.** Let $\boldsymbol{p}_0 = (t_0,\, \tau_0,\, \boldsymbol{a}_0)$ be, as above, an arbitrary point of $\widetilde{\Omega}$. Further, let $I = [c, d]$, $\widetilde{B}$ and $T\colon \mathbf{R}^{n+2} \to \mathbf{R}^{n+2}$ be as above. And as above, let $V = T((c-\tau_0, d-\tau_0) \times \widetilde{B}$. As we observed V is open in $\mathbf{R}^{n+2}$, contains $\boldsymbol{p}_o$ and lies entirely in $\widetilde{\Omega}$. Let $\boldsymbol{G}\colon \widetilde{I} \times \widetilde{B} \longrightarrow \mathbf{R}^{n+1}$ be the map given by

$$\boldsymbol{G}(t,\, \tau,\, \boldsymbol{a}) = \boldsymbol{\psi}_{(\tau,\,\boldsymbol{a})}(t).$$

We know that $\boldsymbol{G}$ is $\mathscr{C}^1$ (see, for instance, the last line of the discussion in §§1.1.2 above, with the role of τ_0 being played by 0). By (2.1.3) we see that

$$\boldsymbol{G}(t,\,\tau,\,\boldsymbol{a}) = (t+\tau,\,\boldsymbol{\varphi}_{(\tau,\,\boldsymbol{a})}(t+\tau)) \qquad (t,\,\tau,\,\boldsymbol{a}) \in \widetilde{I}\times\widetilde{B}. \tag{2.2.1}$$

Formally, we have a map $\boldsymbol{F}\colon\widetilde{\Omega}\to\mathbf{R}^n$ given by

$$\boldsymbol{F}(t,\tau,\boldsymbol{x}) = \boldsymbol{\varphi}_{(\tau,\boldsymbol{x})}(t). \tag{2.2.2}$$

By (2.2.1) we get

$$(s,\boldsymbol{\varphi}_{(\tau,\,\boldsymbol{a})}(s)) = \boldsymbol{G}(s-\tau,\tau,\boldsymbol{a}) \qquad (s,\,\tau,\,\boldsymbol{a})\in V. \tag{2.2.3}$$

Now the map $(s,\,\tau,\,\boldsymbol{a})\mapsto\boldsymbol{G}(s-\tau,\tau,\boldsymbol{a})$ on V is equal to the map $(\boldsymbol{G}\circ T^{-1})|_V$. Since T^{-1} is $\mathscr{C}^\infty$ and $\boldsymbol{G}$ is $\mathscr{C}^1$, it follows from (2.2.3) that the map $(s,\,\tau,\,\boldsymbol{a})\mapsto(s,\boldsymbol{\varphi}_{(\tau,\,\boldsymbol{a})}(s))$ on V is $\mathscr{C}^1$.

In summary, according to the discussion above, if $\boldsymbol{p}_0$ is a point in $\widetilde{\Omega}$, the we can find an open neighbourhood V of $\boldsymbol{p}_0$ in $\widetilde{\Omega}$ such that the map $\boldsymbol{F}\big|_V$ is $\mathscr{C}^1$. We have thus proved the following.

Theorem 2.2.4. *Let $\widetilde{\Omega}$ be the open subset of $\mathbf{R}^{n+2}$ defined in (2.1.1)*[2] *and let $\boldsymbol{F}\colon\widetilde{\Omega}\to\mathbf{R}^n$ be the map given by the formula $\boldsymbol{F}(t,\,\tau,\,\boldsymbol{x}) = \varphi_{(\tau,\,\boldsymbol{x})}(t)$, $(t,\,\tau,\,\boldsymbol{x})\in\widetilde{\Omega}$. Then $\boldsymbol{F}$ is $\mathscr{C}^1$.*

2.2.5. Since being $\mathscr{C}^1$ is a local property, it is clear if we assume $\boldsymbol{v}$ is $\mathscr{C}^1$ and drop the additional hypothesis that it is Lipschitz in $\boldsymbol{x}$, Theorem 2.2.4 remains true. This is because if $\boldsymbol{v}$ is $\mathscr{C}^1$ it is locally Lipschitz in $\boldsymbol{x}$.

3. One parameter groups again

Our results from the last few lectures allow us to make the following observations.

1. All our results transfer to manifolds. So suppose M is a differentiable manifold. For simplicity, assume it is a $\mathscr{C}^\infty$ manifold. Let Ω be an open subset of $\mathbf{R}\times M$ and suppose $v\colon\Omega\to T(M)$ is a continuous function such that

$$\begin{tikzcd} & T(M) \arrow[d, "\varpi"] \\ \Omega \arrow[ur, "v"] \arrow[r, "\text{projection}"'] & M \end{tikzcd}$$

commutes. If M has an atlas such that on each coordinate chart v is locally Lipschitz in the second variable, the the solutions of $\dot{x} = v(t,\,x)$, $x(\tau) = a$ vary continuously with $(t,\,\tau,a)$ for $(\tau,\,a)\in\Omega$ and $t\in J(\tau,a)$, where $J(\tau,a)$ is the maximal interval of existence of the solution $\varphi_{(\tau,a)}$ of the IVP just mentioned. Moreover, if v is $\mathscr{C}^1$, then the set $\widetilde{\Omega} = \{(t,\,\tau,\,x)\in\mathbf{R}\times\Omega \mid t\in J(\tau,\,x)\}$ is open in $\mathbf{R}\times\Omega$, and the map $F\colon\widetilde{\Omega}\to M$ given by $F(t,\,\tau,\,x) = \varphi_{(\tau,\,x)}(t)$ is $\mathscr{C}^1$.

[2]See also Proposition 2.1.4.

2. With M as above, suppose $v\colon T(M)$ is a $\mathscr{C}^1$ vector field such that the intervals of existence $J(0,\,x)$ are all $\mathbf{R}$ as x varies over M. For each $t \in \mathbf{R}$ let $g^t\colon M \to M$ be the map $t \mapsto \varphi_{(0,\,x)}(t)$. the results of Problems 3 and 4 of Homework 6 obviously generalise to give that $\{g^t\}$ is a one parameter group of diffeomorphisms. For fixed $x \in M$, the path $t \mapsto g^t x$ is an integral curve (or phase curve) of the vector field v through x.

3. In particular, if M is a compact $\mathscr{C}^\infty$ manifold and v is a $\mathscr{C}^1$ vector field, then we have a one parameter group of differomorphisms $\{g^t\}$ whose phase velocity field is v.

4. On $GL(n,\,\mathbf{R})$ if $\boldsymbol{v}$ is the vector field defined in Problem 8 of the mid-term exam, then we have a one parameter group in $GL(n,\,\mathbf{R})$ whose phase velocity is $\boldsymbol{v}$. This fact can also be seen by using the properties of the one parameter group $\{e^{tA}\}$ on $GL(n,\,\mathbf{R})$. However the technique outlined here and in the hint in the exam generalises (easily) to Lie groups and gives us the definition of exponential maps on Lie groups. We will not say more about this. Read about this when you have time.

References

[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, Third Edition, translated by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.

[CL] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.