LECTURES 21 AND 22

Dates of the Lectures: March 24 and 29, 2021

As always, $\mathbf{K} \in \{\mathbf{R},\, \mathbf{C}\}$.

The symbol is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \ldots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\|\ \|_2$ and we will simply denote it as $\|\ \|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\|\ \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n,\, \mathbf{K}^n)$.

Note that $(x_1, \ldots, x_n) \neq [x_1 \ldots x_n]$. Each side is the transpose of the other.

1. Preliminaries

Today's lecture is an expansion of the material found in the first chapter of [CL].

1.1. **The basic setting.** We use the notations of Lecture 19. As in that lecture, $\boldsymbol{v} \colon \Omega \to \mathbf{R}^n$ is a Lipschitz continuous function with Lipschitz constant L, with Ω an open subset of $\mathbf{R} \times \mathbf{R}^n$, and (Δ) the associated differential equation

$$(\Delta) \qquad \dot{\boldsymbol{x}} = \boldsymbol{v}(t,\, \boldsymbol{x}).$$

For a point $\boldsymbol{\xi} = (\tau, \boldsymbol{a})$ in Ω, $(\Delta)_{\boldsymbol{\xi}}$ denotes the IVP

$$(\Delta)_{\boldsymbol{\xi}} \qquad \dot{\boldsymbol{x}} = \boldsymbol{v}(t,\, \boldsymbol{x}), \qquad \boldsymbol{x}(\tau) = \boldsymbol{a}.$$

The solution of $(\Delta)_{\boldsymbol{\xi}}$ is denoted $\boldsymbol{\varphi}_{\boldsymbol{\xi}}$, and its maximal interval of existence is denoted $J(\boldsymbol{\xi})$.

[1] See §§2.1 of Lecture 5 of ANA2.

1.2. **More notations and conventions.** Let us fix a point $\boldsymbol{\xi_0} = (\tau_0, \boldsymbol{a_0})$ in Ω for this lecture as well as an interval of existence $I = [\tau_0 - c, \tau_0 + c]$ for $(\Delta)_{\boldsymbol{\xi_0}}$. Set

$$\boldsymbol{\varphi_0} := \boldsymbol{\varphi}_{\boldsymbol{\xi_0}}.$$

Let U_δ be the set of points $\boldsymbol{\xi} = (t, \boldsymbol{a} + \boldsymbol{\varphi_0}(t))$ in $I \times \mathbf{R}^n$ such that $\|\boldsymbol{a}\| < \delta$. We showed in Lemma 3.1.2 of Lecture 19 that there exists a number δ_1 such that the closure $\overline{U}_{\delta_1}$ of U_{δ_1} in $\mathbf{R}^{n+1}$ is a compact subset of Ω. Define δ_m by the formula

$$\delta_m = e^{-L(2c)}\delta_1. \tag{1.2.1}$$

2. Topological Straightening

2.1. The importance of the number δ_m is the following. If $\boldsymbol{a} \in \mathbf{R}^n$ is such that $\|\boldsymbol{a} - \boldsymbol{a_0}\| < \delta_m$, then $(\tau_0, \boldsymbol{a}) \in U_{\delta_m} \subset U_{\delta_1} \subset \Omega$ and by 2.1.6 (b) of Lecture 20, I is an interval of existence for $\boldsymbol{\varphi}_{(\tau_0,\boldsymbol{a})}$. By the *fundamental estimate* it is straightforward to see that $(t, \boldsymbol{\varphi}_{(\tau_0,\boldsymbol{a})}(t)) \in U_{\delta_1}$ for all $t \in I$. We therefore have a map

$$\boldsymbol{\Phi} \colon I \times B(\boldsymbol{a_0}, \delta_m) \to U_{\delta_1} \tag{2.1.1}$$

given by

$$\boldsymbol{\Phi}(t, \boldsymbol{a}) = (t, \boldsymbol{\varphi}_{(\tau_0,\boldsymbol{a})}(t)). \tag{2.1.2}$$

We observed in the second line of this subsection that $\{\tau_0\} \times B(0, \delta_m)$ is contained in U_{δ_m}. Let $S = I \times (\{\tau_0\} \times B(0, \delta_m))$. If, as in the proof of Theorem 2.1.6 of Lecture 20, we define

$$\boldsymbol{F} \colon I \times U_{\delta_m} \to \mathbf{R}^n \tag{2.1.3}$$

as the map $(t, \tau, \boldsymbol{a}) \mapsto \boldsymbol{\varphi}_{(\tau,\boldsymbol{a})}(t)$, then we see that $\boldsymbol{\Phi}$ is the graph of the restriction $\boldsymbol{F}|_S$, and hence is continuous by *loc.cit.* More explcitly, we have

$$\boldsymbol{\Phi}(t, \boldsymbol{a}) = (t, \boldsymbol{F}(t, \tau_0, \boldsymbol{a})) \tag{2.1.4}$$

establishing the continuity of $\boldsymbol{\Phi}$.

A few things are worth observing:

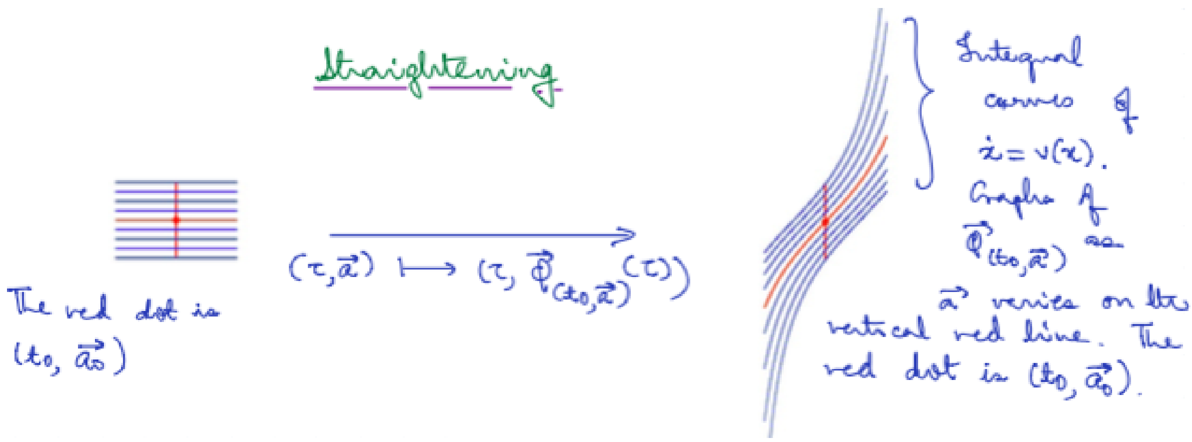
(i) By the uniqueness of solutions to $(\Delta)_{\boldsymbol{\xi}}$ it is clear that $\boldsymbol{\Phi}$ is one-to-one.

(ii) Let $R_\delta^\circ = (\tau_0 - c, \tau_0 + c) \times B(\boldsymbol{a_0}, \delta)$, $R_\delta = I \times B(\boldsymbol{a_0}, \delta)$, and $\overline{R}_\delta = I \times \overline{B}(\boldsymbol{a_0}, \delta)$. For $\delta \in (0, \delta_m)$, set

$$V_\delta^\circ = \boldsymbol{\Phi}(R_\delta^\circ), \quad V_\delta = \boldsymbol{\Phi}(R_\delta), \quad \text{and} \quad \overline{V}_\delta = \boldsymbol{\Phi}(\overline{R}_\delta). \tag{2.1.5}$$

Since $\boldsymbol{\Phi}$ is one-to-one and continuous we see by L. E. J. Brouwer's theorem on *the invariance of domain* for $\mathbf{R}^{n+1}$ that V_δ° is an open subset of $\mathbf{R}^{n+1}$ and $\boldsymbol{\Phi}|_{R_\delta^\circ}$ gives a homeomorphism between R_δ° and V_δ°. Further, one again using the fact that $\boldsymbol{\Phi}$ is injective, since $\overline{R}_\delta$ is compact , the restriction $\boldsymbol{\Phi}|_{\overline{R}_\delta}$ gives a homeomorphism between $\overline{R}_\delta$ and $\overline{V}_\delta$. It follows that $\boldsymbol{\Phi}$ also induces a homeomorphism between R_δ and V_δ, and V_δ° is the interior of $\overline{V}_\delta$ in $\mathbf{R}^{n+1}$.

The following picture summarises the situation.



3. Differentiability with respect to the initial phase

In this section we assume the continuous function is $\boldsymbol{v}$ is $\mathscr{C}^1$ in $\boldsymbol{x}$. By this we mean that the partial derivative of $\frac{\partial \boldsymbol{v}}{\partial x_j}$ exist on Ω and are continuous in $(t, \boldsymbol{x})$. Note that this implies $\boldsymbol{v}$ is locally Lipschitz in $\boldsymbol{x}$.

3.1. The Equation of Variations.
Let

$$D_2(t, \boldsymbol{x}) = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial v_n}{\partial x_1} & \cdots & \frac{\partial v_n}{\partial x_n} \end{bmatrix} \tag{3.1.1}$$

Recall the rectangles R_δ, R_δ°, $\overline{R}_\delta$ defined in §§ **1.1** (see especially (2.1.5)). Define

$$A\colon R_{\delta_m} \longrightarrow M_n(\mathbf{R}) \tag{3.1.2}$$

by the formula

$$A(t, \boldsymbol{x}) := D_2(\boldsymbol{\Phi}(t, \boldsymbol{x})) \qquad ((t, \boldsymbol{x}) \in R_{\delta_m}). \tag{3.1.3}$$

For each fixed t and $\boldsymbol{x}$, $A(t, \boldsymbol{x})$ is a linear transformation on $\mathbf{R}^n$. Consider the *equation of variations*[2] given by

$$\dot{\boldsymbol{z}} = A(t, \boldsymbol{x})\boldsymbol{z}, \quad \boldsymbol{z}(\tau_0) = \mathbf{e}_j. \tag{3.1.4}$$

The differential equation $\dot{\boldsymbol{z}} = A(t, \boldsymbol{x})\boldsymbol{z}$ is really a family of differential equations, one for each $\boldsymbol{x}$ and is *linear* for each fixed $\boldsymbol{x}$. It should be said that the differential equation $\dot{z} = A(t, \boldsymbol{x})z$ is also called the equation of variations, not just the initial value problems in (3.1.4).

Let $\boldsymbol{\zeta}$ be the unique solution of the linear IVP (3.1.4). Since (3.1.4) is linear, we know from general theory that this unique solution exists on I (see Theorem 1.1.1 of Lecture 7). Note that $\boldsymbol{\zeta}$ depends upon $\boldsymbol{x}$, and therefore we think of $\boldsymbol{\zeta}$ as a function of two variables, and write $\boldsymbol{\zeta}(t, \boldsymbol{x})$ for $\zeta(t)$ if we wish to emphasise the role of $\boldsymbol{x}$.

[2] Problem 4 of HW 7 and Problem 9 of the mid-term.

3.2. **Differentiability with respect to $\boldsymbol{x}$.** Since $\overline{U}_{\delta_1}$ is compact (see [Lecture 19, Lemma 3.1.2]) and $\boldsymbol{v}$ is $\mathscr{C}^1$, therefore D_2 is bounded on $\overline{U}_{\delta_1}$. Let $0 < M < \infty$ be such that

$$\|D_2(t, \boldsymbol{a})\|_\circ \le M \qquad ((t, \boldsymbol{a}) \in \overline{U}_{\delta_1}). \tag{3.2.1}$$

Let $\boldsymbol{F}\colon I \times U_{\delta_m} \to \mathbf{R}^n$ be the map defined in (2.1.3). Note that if $\boldsymbol{x}$ and $\boldsymbol{x} + h\mathbf{e}_j$ both lie in $B(\boldsymbol{a_0}, \delta_m)$, then for $t \in (\tau_0 - c, \tau_0 + c)$,

$$\|\boldsymbol{F}(t, \tau_0, \boldsymbol{x} + h\mathbf{e}_j) - \boldsymbol{F}(t, \tau_0, \boldsymbol{x})\| \le |h| e^{2Lc}. \tag{3.2.2}$$

This follows from the fundamental estimate and the fact that $\boldsymbol{\varphi}_{\tau_0, \boldsymbol{a}}$ is an ε-approximate solution of $\dot{\boldsymbol{y}} = \boldsymbol{v}(t, \boldsymbol{y})$ with $\varepsilon = 0$ (set $\boldsymbol{y}(\tau_0) = \boldsymbol{x} + h\mathbf{e}_j$ and $\boldsymbol{y}(\tau_0) = \boldsymbol{x}$ to get two exact solutions and use the fundamental estimate).

Here is the main theorem we are interested in.

Theorem 3.2.3. *Suppose, as above, $\boldsymbol{v}$ is $\mathscr{C}^1$ in $\boldsymbol{x}$. Let $\tau_0 \in I$ be a fixed initial time poiint. Then $\boldsymbol{F}(t, \tau_0, \boldsymbol{x})$ is $\mathscr{C}^1$ as a function of $(t, \boldsymbol{x})$ on $R^\circ_{\delta_m}$.*

Remark. Let us write $\boldsymbol{F}(t, \boldsymbol{x})$ for $\boldsymbol{F}(t, \tau_0, \boldsymbol{x})$. Suppose the the theorem is true. The identity $\boldsymbol{F}(t, \boldsymbol{x}) = \boldsymbol{\varphi}_{(\tau_0, \boldsymbol{x})}(t)$ yields the identity $\frac{\partial \boldsymbol{F}}{\partial t}\Big|_{(t, \boldsymbol{x})} = \boldsymbol{v}(t, \boldsymbol{F}(t, \boldsymbol{x})) = \boldsymbol{v}(\boldsymbol{\Phi}(t, \boldsymbol{x}))$, which in turn means that $\frac{\partial \boldsymbol{F}}{\partial t}$ has partial derivatives with respect to $\boldsymbol{x}$ which are continuous in $(t, \boldsymbol{x})$, for $\boldsymbol{v}$ is $\mathscr{C}^1$ in $\boldsymbol{x}$. In other words, the mixed partial $\frac{\partial^2 \boldsymbol{F}}{\partial x_j \partial t}$ exists and is continuous in $(t, \boldsymbol{x})$. By Theorem 2.1.3 of Lecture 11 of ANA2 we then conclude that $\frac{\partial^2 \boldsymbol{F}}{\partial t \partial x_j}$ also exists and

$$\frac{\partial^2 \boldsymbol{F}}{\partial t \partial x_j} = \frac{\partial^2 \boldsymbol{F}}{\partial x_j \partial t}.$$

Via the chain rule and (2.1.4) we then have a sequence of equalities:

$$\frac{\partial^2 \boldsymbol{F}}{\partial t \partial x_j}(t, \boldsymbol{x}) = \frac{\partial^2 \boldsymbol{F}}{\partial x_j \partial t}(t, \boldsymbol{x}) = \frac{\partial}{\partial x_j} \boldsymbol{v}(\boldsymbol{\Phi}(t, \boldsymbol{x})) = D_2(\boldsymbol{\Phi}(t, \boldsymbol{x})) \frac{\partial \boldsymbol{F}}{\partial x_j} = A(t, \boldsymbol{x}) \frac{\partial \boldsymbol{F}}{\partial x_j}.$$

Thus each of the partial derivatives $\frac{\partial \boldsymbol{F}}{\partial x_j}$ satisfies the equation of variations $\dot{z} = A(t, \boldsymbol{x}) z$. Moreover, since $F(\tau_0, \boldsymbol{x}) = \boldsymbol{x}$, it follows that $\frac{\partial \boldsymbol{F}}{\partial x_j}\Big|_{(\tau_0, \boldsymbol{x})} = \mathbf{e}_j$. In conclusion, if $\boldsymbol{F}$ is differentiable in the phase variables, and the resulting partial derivatives are continuous in $(t, \boldsymbol{x})$, then $\frac{\partial \boldsymbol{F}}{\partial x_j}$ satisfies the IVP (3.1.4). Thus, if we have to search for the j^{th} partial derivative of $\boldsymbol{F}$ with respect to the phase variables, we have no choice but to look at the solution to the equation of variations (3.1.4). Indeed our argument above shows that if the solution to (3.1.4) is not $\frac{\partial \boldsymbol{F}}{\partial x_j}$, then $\frac{\partial \boldsymbol{F}}{\partial x_j}$ does not exist as a continuous function of $(t, \boldsymbol{x})$.

Proof of Theorem 3.2.3. As above, since τ_0 is fixed throughout this proof, to reduce clutter let us suppress τ_0 in the notation for $\boldsymbol{F}$ and write

$$\boldsymbol{F}(t, \boldsymbol{x}) = \boldsymbol{F}(t, \tau_0, \boldsymbol{x}).$$

Let $(t, \boldsymbol{x}) \in R^\circ_{\delta_m}$. We wish to show the partial derivatives $\frac{\partial \boldsymbol{F}}{\partial x_j}$, $j = 1, \dots, n$, exist at $(t, \boldsymbol{x})$ and that they are continuous in $(t, \boldsymbol{x})$. We already know the analogous statement for $\frac{\partial \boldsymbol{F}}{\partial t}$, since this is equal to $\boldsymbol{v}(t, \boldsymbol{F}(t, \boldsymbol{x}))$ which is continuous in $(t, \boldsymbol{x})$.

We can find $0 < \delta < \delta_m$ such that $(t, \boldsymbol{x}) \in R_\delta^\circ \subset R_{\delta_m}^\circ$. Let V_δ, V_δ°, and $\overline{V}_\delta$ be as in (2.1.5). We have

$$V_\delta^\circ \subset V_\delta \subset \overline{V}_\delta$$

with $\overline{V}_\delta$ compact and V_δ° open in $\mathbf{R}^{n+1}$.

For $\eta > 0$, the *modulus of continuity* $\omega(\eta)$ of D_2 on $\overline{V}_\delta$ is defined to be:

$$\omega(\eta) = \sup\Big\{\|D_2(t, \boldsymbol{x}_1) - D_2(t, \boldsymbol{x}_2)\|_\circ \,\Big|\, (t, \boldsymbol{x}_i) \in \overline{V}_\delta,\ i = 1, 2, \|\boldsymbol{x_1} - \boldsymbol{x_2}\| \leq \eta\Big\}.$$

Since D_2 is continuous on the compact set $\overline{V}_\delta$, it is uniformly continuous on $\overline{V}_\delta$, and so $\omega(\eta) \to 0$ as $\eta \to 0$. By the nature of supremums, $\omega(\eta)$ is a increasing function of η. We claim that

$$\Big\|\boldsymbol{v}(t, \boldsymbol{x}_2) - \boldsymbol{v}(t, \boldsymbol{x}_1) - D_2(t, \boldsymbol{x}_1)(\boldsymbol{x}_2 - \boldsymbol{x}_1)\Big\| \leq \omega(\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \tag{3.2.3.1}$$

for $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ such that the line segment joining $(t, \boldsymbol{x}_1)$ and $(t, \boldsymbol{x}_2)$ lies in $\overline{V}_\delta$. Indeed, in this case we have (by the chain rule in several varaiables)

$$\frac{d}{ds}\boldsymbol{v}(t, \boldsymbol{x}_1 + s(\boldsymbol{x}_2 - \boldsymbol{x}_1)) = D_2(t, \boldsymbol{x}_1 + s(\boldsymbol{x}_2 - \boldsymbol{x}_1))(\boldsymbol{x}_2 - \boldsymbol{x}_1)$$

whence,

$$\boldsymbol{v}(t, \boldsymbol{x}_2) - \boldsymbol{v}(t, \boldsymbol{x}_1) = \int_0^1 \Big(D_2(t, \boldsymbol{x}_1 + s(\boldsymbol{x}_2 - \boldsymbol{x}_1))\Big)(\boldsymbol{x}_2 - \boldsymbol{x}_1)ds. \tag{3.2.3.2}$$

Let

$$I = \int_0^1 \Big(D_2(t, \boldsymbol{x}_1 + s(\boldsymbol{x}_2 - \boldsymbol{x}_1)) - D_2(t, \boldsymbol{x}_1)\Big)(\boldsymbol{x}_2 - \boldsymbol{x}_1)ds.$$

We note two things

(i) $\|I\| \leq \omega(\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|)\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$, and

(ii) $I = \boldsymbol{v}(t, \boldsymbol{x}_2) - \boldsymbol{v}(t, \boldsymbol{x}_1) - D_2(t, \boldsymbol{x}_1)(\boldsymbol{x}_2 - \boldsymbol{x}_1)$.

We are using the equality $\int_0^1 D_2(t, \boldsymbol{x}_1)(\boldsymbol{x}_2 - \boldsymbol{x}_1)ds = D_2(t, \boldsymbol{x}_1)(\boldsymbol{x}_2 - \boldsymbol{x}_1)$ as well as the equality (3.2.3.2) for (ii). The inequality (3.2.3.1) follows.

Consider the difference quotient

$$\boldsymbol{\theta}(t, \boldsymbol{x}, h) = \frac{\boldsymbol{F}(t, \boldsymbol{x} + h\mathbf{e}_j) - \boldsymbol{F}(t, \boldsymbol{x})}{h}.$$

Now $(t, \boldsymbol{x})$ is in the interior of R_δ, i.e. in R_δ°, hence $\boldsymbol{\Phi}(t, \boldsymbol{x}) \in V_\delta^\circ$, which we argued (via the invariance of domain theorem of L. E. J. Brouwer) is open in $\mathbf{R}^{n+1}$. Therefore, for small values of $|h|$, the line segment connecting $\boldsymbol{\Phi}(t, \boldsymbol{x})$ and $\boldsymbol{\Phi}(t, \boldsymbol{x} + h\mathbf{e}_j)$ lies entirely in $V_\delta^\circ \subset \overline{V}_\delta$. For such h we have

$$\Big\|\dot{\boldsymbol{\theta}}(t, \boldsymbol{x}, h) - A(t, \boldsymbol{x})\boldsymbol{\theta}(t, \boldsymbol{x}, h)\Big\| = \frac{1}{|h|}\|\boldsymbol{q}\|$$

where

$$\boldsymbol{q} = \boldsymbol{v}(t, \boldsymbol{F}(t, \boldsymbol{x} + h\mathbf{e}_j)) - \boldsymbol{v}(t, \boldsymbol{F}(t, \boldsymbol{x})) - D_2(t, \boldsymbol{F}(t, \boldsymbol{x}))(\boldsymbol{F}(t, \boldsymbol{x} + h\mathbf{e}_j) - \boldsymbol{F}(t, \boldsymbol{x}))$$

By (3.2.3.1), this means, with $\eta = \|\boldsymbol{F}(t, \boldsymbol{x} + h\mathbf{e}_j) - \boldsymbol{F}(t, \boldsymbol{x})\|$, we have

$$\Big\|\dot{\boldsymbol{\theta}}(t, \boldsymbol{x}, h) - A(t, \boldsymbol{x})\boldsymbol{\theta}(t, \boldsymbol{x}, h)\Big\| \leq \frac{1}{|h|}\omega(\eta)\eta.$$

By (3.2.2), $\eta \leq |h|e^{2Lc}$. Since ω is an increasing function, we conclude that

$$\Big\|\dot{\boldsymbol{\theta}}(t, \boldsymbol{x}, h) - A(t, \boldsymbol{x})\boldsymbol{\theta}(t, \boldsymbol{x}, h)\Big\| \leq \omega(|h|e^{2Lc})e^{2Lc}. \tag{3.2.3.3}$$

Set $\varepsilon(h) = \omega(|h|e^{2Lc})e^{2Lc}$. According to (3.2.3.3), $\boldsymbol{\theta}(-, \boldsymbol{x}, h)$ is an $\varepsilon(h)$-approximate solution of the equation of variations (3.1.4).

As before, let $\boldsymbol{\zeta}$ be the solution of the equation of variations (3.1.4). Since M as in (3.2.1) is a Lipschitz constant for the linear differential equation $\dot{\boldsymbol{z}} = A(t, \boldsymbol{x})\boldsymbol{z}$, and since $\boldsymbol{\zeta}(\tau_0) = \boldsymbol{\theta}(\tau_0, \boldsymbol{x}, h) = \mathbf{e}_j$, therefore the fundamental estimate (with $\varepsilon_1 = \varepsilon(h)$ and $\varepsilon_2 = 0$) gives

$$\|\boldsymbol{\theta}(t, \boldsymbol{x}, h) - \boldsymbol{\zeta}(t, \boldsymbol{x})\| \leq \frac{\varepsilon(h)}{M}\left(e^{2Mc} - 1\right) \quad (t \in (\tau_0 - c, \tau_0 + c)).$$

Now $\varepsilon(h) \to 0$ as $h \to 0$. The above inequality then implies that the limit of $\boldsymbol{\theta}(t, \boldsymbol{x}, h)$ as $h \to 0$ exists and $\lim_{h\to 0} \boldsymbol{\theta}(t, \boldsymbol{x}, h) = \boldsymbol{\zeta}(t, \boldsymbol{x})$. This is another way of saying $\left.\frac{\partial \boldsymbol{F}}{\partial x_j}\right|_{(t,\boldsymbol{x})}$ exists and

$$\left.\frac{\partial \boldsymbol{F}}{\partial x_j}\right|_{(t,\boldsymbol{x})} = \boldsymbol{\zeta}(t, \boldsymbol{x}).$$

This proves the existence of partial derivatives of $\boldsymbol{F}$, not their continuity. We already know that $\frac{\partial \boldsymbol{F}}{\partial t}$ is continuous since it equals $\boldsymbol{v}(t, \boldsymbol{F}(t, \boldsymbol{x}))$. So we only have to show that $\frac{\partial \boldsymbol{F}}{\partial x_j}$ is continuous for each $j \in \{1, \ldots, n\}$. This is equivalent to showing that $\boldsymbol{\zeta}(t, \boldsymbol{x})$ is continuous in $(t, \boldsymbol{x})$. To see this, consider the IVP

(3.2.3.4)
$$\begin{bmatrix} \dot{\boldsymbol{z}} \\ \dot{\boldsymbol{y}} \end{bmatrix} = \begin{bmatrix} A(t, \boldsymbol{y})\boldsymbol{z} \\ 0 \end{bmatrix}; \qquad \begin{bmatrix} \boldsymbol{z}(\tau_0) \\ \boldsymbol{y}(\tau_0) \end{bmatrix} = \begin{bmatrix} \mathbf{e}_j \\ \boldsymbol{x} \end{bmatrix}.$$

Check that $\left(t, \left[\begin{smallmatrix} \boldsymbol{z} \\ \boldsymbol{y} \end{smallmatrix}\right]\right) \mapsto \left[\begin{smallmatrix} A(t,\boldsymbol{y})\boldsymbol{z} \\ 0 \end{smallmatrix}\right]$ is Lipschitz in $\left[\begin{smallmatrix} \boldsymbol{z} \\ \boldsymbol{y} \end{smallmatrix}\right]$. The solution of the IVP (3.2.3.4) is clearly

$$t \mapsto \begin{bmatrix} \boldsymbol{\zeta}(t, \boldsymbol{x}) \\ \boldsymbol{x} \end{bmatrix} \qquad (t \in I).$$

By Theorem 2.1.6 of Lecture 20 this solution is jointly continuous in the time variable t and the initial phase $\left[\begin{smallmatrix} \boldsymbol{e}_j \\ \boldsymbol{x} \end{smallmatrix}\right]$. It follows that $\boldsymbol{\zeta}$ is continuous in $(t, \boldsymbol{x})$ as required. □

References

[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, Third Edition, translated by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.

[CL] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.