LECTURE 20

Date of the Lecture: March 22, 2021

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \, \|_2$ and we will simply denote it as $\| \, \|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\| \, \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n, \mathbf{K}^n)$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. Basic notations and conventions

1.1. Throughout this lecture we assume that $\boldsymbol{v} \colon \Omega \to \mathbf{R}^n$ is a continuous function which is Lipschitz in $\boldsymbol{x}$ with Lipschitz constant L on a domain[2] Ω contained in $\mathbf{R} \times \mathbf{R}^n$. Our interest is in the behaviour of solutions of the differential equation

$$(\Delta) \qquad \dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$$

as we vary the initial conditions, i.e. the initial time point and the initial state. To that end, for a point $\boldsymbol{\xi} = (\tau, \boldsymbol{a})$ in Ω, the symbol $\boldsymbol{\varphi}_{(\tau, \boldsymbol{a})}$ (or simply $\boldsymbol{\varphi}_{\boldsymbol{\xi}}$) will denote the unique solution to initial value problem $(\Delta)_{(\tau, \boldsymbol{a})}$ below.

$$(\Delta)_{(\tau, \boldsymbol{a})} \qquad \dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x}), \quad \boldsymbol{x}(\tau) = \boldsymbol{a}.$$

We might often write $(\Delta)_{\boldsymbol{\xi}}$ for $(\Delta)_{(\tau, \boldsymbol{a})}$. For $\boldsymbol{\xi} = (\tau, \boldsymbol{a}) \in \Omega$, $J(\boldsymbol{\xi}) = J(\tau, \boldsymbol{a})$ will denote the maximal interval of existence for $(\Delta)_{(\tau, \boldsymbol{a})}$.

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[1]See §§2.1 of Lecture 5 of ANA2.

[2]i.e. a non-empty connected open subset of $\mathbf{R}^n$

2. **Continuity with respect to initial conditions**

We know from the remark in §§§ 1.1.2 of Lecture 6 as well as Theorem 1.1.2 (a) of Lectures 17 and 18 (when $\boldsymbol{v}$ is $\mathscr{C}^1$) that given a point $\boldsymbol{\xi_0} = (\tau_0, \boldsymbol{a_0}) \in \Omega$, we have an open neighbourhood W of $\boldsymbol{\xi_0}$ in Ω and a positive number b such that for every $\boldsymbol{\xi} = (\tau, \boldsymbol{a}) \in W$, $(\tau - b, \tau + b)$ is an interval of existence for the IVP $(\Delta)_{\boldsymbol{\xi}}$. By shrinking W if necessary, we may assume $W = (\tau_0 - \varepsilon, \tau_0 + \varepsilon) \times B(\boldsymbol{a_0}, r)$ with $\varepsilon < b/2$. In such a case, we have a map $\boldsymbol{F}\colon (\tau_0 - \varepsilon, \tau_0 + \varepsilon) \times W \to \mathbf{R}^n$ given by $\boldsymbol{F}(t, \boldsymbol{\xi}) = \boldsymbol{\varphi_\xi}(t)$. One can ask if this map is continuous, or if it is $\mathscr{C}^1$. In this section we will prove that $\boldsymbol{F}$ is continuous. Later we will show that it is $\mathscr{C}^1$ if $\boldsymbol{v}$ is $\mathscr{C}^1$. We remind the reader that in this lecture $\boldsymbol{v}$ is Lipschitz in $\boldsymbol{x}$.

2.1. For the rest of this section, we fix a solution $\boldsymbol{\varphi}\colon [c,\, d] \to \Omega$ of the differential equation (Δ). As in (3.1.1) of Lecture 19 for each $\delta > 0$, set

$$U_\delta = \big\{(\tau, \boldsymbol{a}) \in \mathbf{R}^{n+1} \mid \tau \in [c, d],\ \|\boldsymbol{a} - \boldsymbol{\varphi}(\tau)\| < \delta\big\}. \tag{2.1.1}$$

There are other descriptions of U_δ. Here are two more

$$U_\delta = \big\{(\tau, \boldsymbol{a} + \boldsymbol{\varphi}(\tau)) \in \mathbf{R}^{n+1} \mid \tau \in [c, d],\ \|\boldsymbol{a}\| < \delta\big\}. \tag{2.1.2}$$

Finally, if $f\colon [c, d] \times \mathbf{R}^n \to [c, d] \times \mathbf{R}^n$ is the homeomorphism $(t, \boldsymbol{x}) \mapsto (t, \boldsymbol{x} + \boldsymbol{\varphi}(t))$, and R_δ the rectangle $[c, d] \times B(\mathbf{0},\, \delta)$, then

$$U_\delta = f(R_\delta). \tag{2.1.3}$$

Note that the closure of R_δ in $[c, d] \times \mathbf{R}^n$ is $\overline{R}_\delta := [c, d] \times \overline{B}(\mathbf{0},\, \delta)$, whence, since f is a homeomorphism,

$$\overline{U}_\delta = f(\overline{R}_\delta). \tag{2.1.4}$$

2.1.5. According to [Lecture 19, Lemma 3.1.2] there is a δ_1 such that the closure $\overline{U}_{\delta_1}$ of U_{δ_1} in $\mathbf{R}^n$ is a compact subset of Ω. We will show that there is a δ with $0 < \delta < \delta_1$ such that $[c, d]$ is an interval of existence for $(\Delta)_{\boldsymbol{\xi}}$ for every $\boldsymbol{\xi} = (\tau,\, \boldsymbol{a})$ in U_δ. Note that this is not the same as the assertion in 1.1.2 of Lecture 6 though it is closely related to that.

Theorem 2.1.6. *With the above notations, there exists a $\delta > 0$, such that*

(a) $U_\delta \subset \Omega$.
(b) *For every $\boldsymbol{\xi} = (\tau, \boldsymbol{a}) \in U_\delta$ the solution $\boldsymbol{\varphi_\xi}$ to $(\Delta)_{\boldsymbol{\xi}}$ exists on $[c, d]$.*
(c) *The map $(t, \tau, \boldsymbol{a}) \mapsto \boldsymbol{\varphi}_{(\tau, \boldsymbol{a})}(t)$ is uniformly continuous on $V = [c, d] \times U_\delta$.*

Proof. Fix $\delta_1 > 0$ as in 2.1.5, i.e. fix δ_1 such that $\overline{U}_{\delta_1}$ is a compact subset in Ω. Let $\mathbf{D} = \{\delta \mid 0 < \delta < e^{-L(d-c)}\delta_1\}$. We will show that (a), (b), and (c) are true for every δ in $\mathbf{D}$.

Let

$$U = \bigcup_{\delta \in \mathbf{D}} U_\delta.$$

For $\boldsymbol{\xi} \in U$, the fundamental estimate (with $\epsilon_1 = \epsilon_2 = 0$) yields

$$\|\boldsymbol{\varphi}(t) - \boldsymbol{\varphi_\xi}(t)\| < \delta_1$$

for all $t \in [c, d] \cap J(\boldsymbol{\xi})$, where, as agreed upon, $J(\boldsymbol{\xi})$ is the maximal interval of existence associated with $(\Delta)_{\boldsymbol{\xi}}$. In other words, $(t, \boldsymbol{\varphi_\xi}(t)) \in U_{\delta_1}$ for all t in the intersection $J(\boldsymbol{\xi}) \cap [c, d]$. Since $(t, \boldsymbol{\varphi_\xi}(t))$ must exit the compact set $\overline{U}_{\delta_1}$, the above

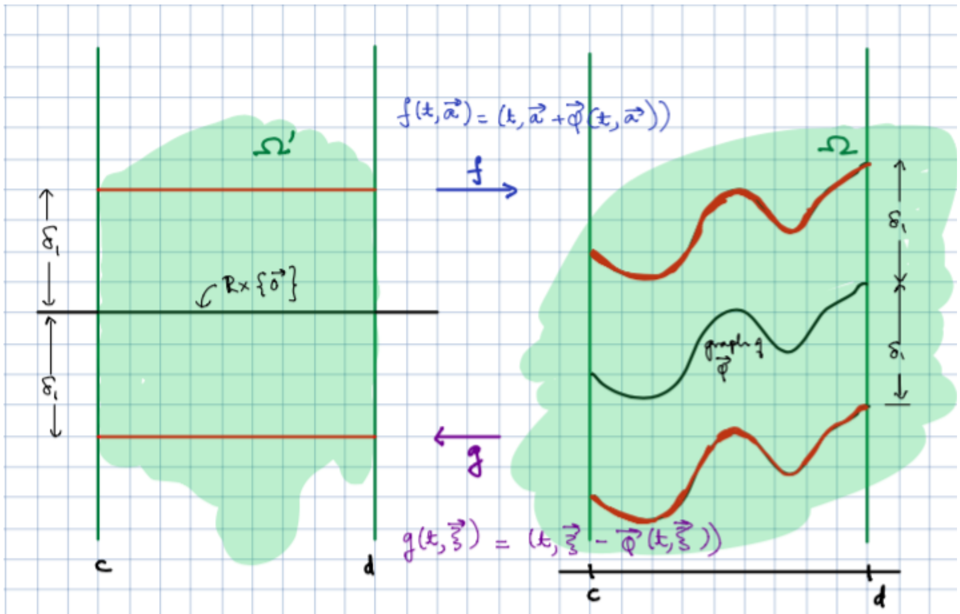

FIGURE 1. U_{δ_1} is the region between the two brown curves in the figure on the right

inequality forces it to exit at $\{c\} \times \mathbf{R}^n$ and $\{d\} \times \mathbf{R}^n$. Thus $[c, d] \cap J(\boldsymbol{\xi}) = [c, d]$, i.e. $[c, d]$ is an interval of existence for $\boldsymbol{\xi}$ for every $\boldsymbol{\xi} \in U$.

Let $\boldsymbol{F} \colon [c, d] \times U \to \mathbf{R}^n$ be the map given by the formula

$$\boldsymbol{F}(t, \tau, \boldsymbol{a}) = \boldsymbol{\varphi}_{(\tau, \boldsymbol{a})}(t)$$

for $(t, \tau, \boldsymbol{a}) \in [c, d] \times U$. Since U is a subset of $\overline{U}_{\delta_1}$ which is compact, $\boldsymbol{v}$ is bounded in U. Let $M < \infty$ be an upper bound for $\|\boldsymbol{v}\|$ in U.

Let $\boldsymbol{\xi_0} = (\tau_0, \boldsymbol{a_0}) \in U$ and let us examine the continuity of $\boldsymbol{F}$ at $(s, \boldsymbol{\xi_0}) \in [c, d] \times U$. Since U is open in $[c, d] \times \mathbf{R}^n$, there exists a rectangle $W = [\alpha, \beta] \times B(\boldsymbol{a_0}, r)$ in U containing $\boldsymbol{\xi_0}$, and hence, for every $\boldsymbol{\xi} = (\tau, \boldsymbol{a}) \in W$, the line segment $[\alpha, \beta] \times \{\boldsymbol{a}\}$ lies in $U \subset \Omega$. Applying [Lecture 19, Lemma 2.1.3], we see that

$$(\dagger) \qquad \|\boldsymbol{a} - \boldsymbol{\varphi}_{(\tau, \boldsymbol{a})}(\tau_0)\| \leq \tfrac{M}{L}(e^{L|\tau - \tau_0|} - 1) \qquad (\boldsymbol{\xi} = (\tau, \boldsymbol{a}) \in W).$$

We claim that $\boldsymbol{\varphi_\xi} \to \boldsymbol{\varphi_{\xi_0}}$, uniformly on $[c, d]$, as $\boldsymbol{\xi} \to \boldsymbol{\xi_0}$. We may assume that $\boldsymbol{\xi}$ approaches $\boldsymbol{\xi_0}$ through points in W. By the fundamental estimate (used twice) we have

$$(\ddagger) \qquad \begin{aligned} \|\boldsymbol{\varphi_{\xi_0}}(t) - \boldsymbol{\varphi_\xi}(t)\| &\leq \|\boldsymbol{\varphi_{\xi_0}}(\tau_0) - \boldsymbol{\varphi_\xi}(\tau_0)\| e^{L(d-c)} \\ &= \|\boldsymbol{a_0} - \boldsymbol{\varphi}_{(\tau, \boldsymbol{a})}(\tau_0)\| e^{L(d-c)} \\ &\leq \|\boldsymbol{a_0} - \boldsymbol{a}\| e^{L(d-c)} + \|\boldsymbol{a} - \boldsymbol{\varphi}_{(\tau, \boldsymbol{a})}(\tau_0)\| e^{L(d-c)} \\ &\leq \|\boldsymbol{a_0} - \boldsymbol{a}\| e^{L(d-c)} + \tfrac{M}{L}(e^{L|\tau - \tau_0|} - 1) e^{L(d-c)} \end{aligned}$$

for $\boldsymbol{\xi} \in W$. The last inequality is from $(\dagger)$. The expression in the bottom line, namely $h(\boldsymbol{\xi}) = \|\boldsymbol{a_0} - \boldsymbol{a}\| e^{L(d-c)} + \frac{M}{L}(e^{L|\tau - \tau_0|} - 1) e^{L(d-c)}$, is a continuous function of $\boldsymbol{\xi} = (\tau, \boldsymbol{a})$ which is independent of $t \in [c, d]$. Moreover, $h(\boldsymbol{\xi}) \to 0$ as $\boldsymbol{\xi} \to \boldsymbol{\xi_0}$.

This means $\boldsymbol{\varphi_\xi} \to \boldsymbol{\varphi_{\xi_0}}$, uniformly on $[c, d]$, as $\boldsymbol{\xi} \to \boldsymbol{\xi_0}$. In greater detail, given $\varepsilon > 0$, we can find $\eta > 0$ (independent of $t \in [c, d]$) such that $h(\boldsymbol{\xi}) < \varepsilon$ whenever $\|\boldsymbol{\xi} - \boldsymbol{\xi_0}\| < \eta$. Thus, from (‡), $\|\boldsymbol{\varphi_{\xi_0}}(t) - \boldsymbol{\varphi_\xi}(t)\| < \epsilon$ for all $t \in [c, d]$, whenever $\|\boldsymbol{\xi} - \boldsymbol{\xi_0}\| < \eta$. This proves the assertion of uniform convergence.

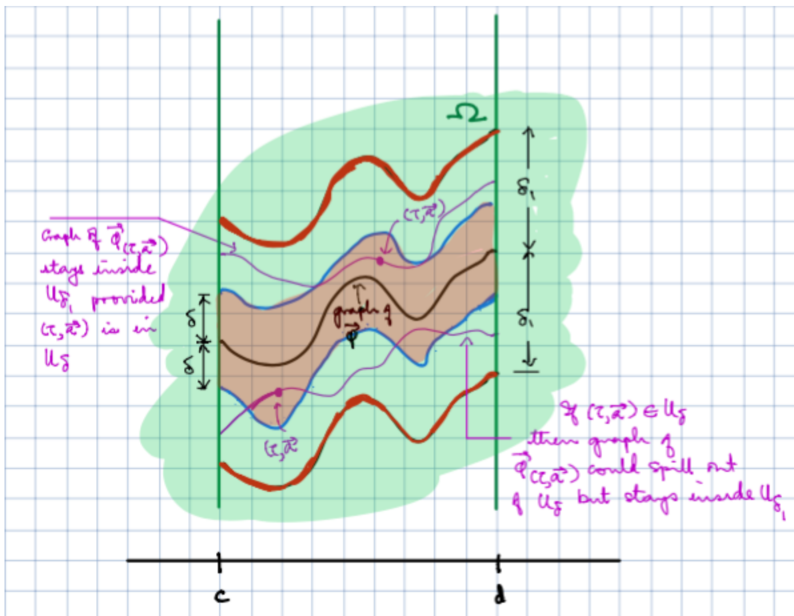

Figure 2.

We now show that $\boldsymbol{F} \colon [c, d] \times U \to \mathbf{R}^n$ is continuous. Let $\boldsymbol{\varphi_0} = \boldsymbol{\varphi_{\xi_0}}$. Since $\boldsymbol{\varphi_\xi}$ converges uniformly on $[c, d]$ to $\boldsymbol{\varphi_0}$ as $\boldsymbol{\xi} \to \boldsymbol{\xi_0}$, therefore given $\varepsilon > 0$ we can find $\eta_1 > 0$ such that

$$\|\boldsymbol{\varphi_\xi}(s) - \boldsymbol{\varphi_0}(s)\| < \varepsilon \quad (s \in [c, d])$$

whenever $\|\boldsymbol{\xi} - \boldsymbol{\xi_0}\| < \eta_1$. Now $\boldsymbol{\varphi_0}$ is uniformly continuous on the compact set $[c, d]$, and hence there exists $\eta_2 > 0$ such that

$$\|\boldsymbol{\varphi_0}(t) - \boldsymbol{\varphi_0}(s)\| < \varepsilon$$

whenever $|t - s| < \eta_2$. Since

$$\|\boldsymbol{F}(t, \boldsymbol{\xi}) - \boldsymbol{F}(s, \boldsymbol{\xi_0})\| = \|\boldsymbol{\varphi_\xi}(t) - \boldsymbol{\varphi_0}(s)\| \leq \|\boldsymbol{\varphi_\xi}(t) - \boldsymbol{\varphi_0}(t)\| + \|\boldsymbol{\varphi_0}(t) - \boldsymbol{\varphi_0}(s)\|,$$

it follows that

$$\|\boldsymbol{F}(t, \boldsymbol{\xi}) - \boldsymbol{F}(s, \boldsymbol{\xi_0})\| < 2\varepsilon$$

whenever $\|\boldsymbol{\xi} - \boldsymbol{\xi_0}\| < \eta_1$ and $|t - s| < \eta_2$. Thus $\boldsymbol{F}$ is continuous on $[c, d] \times U$.

We will now prove (a), (b) and (c) for every $\delta \in \mathbf{D}$, i.e. for every δ such that $0 < \delta < e^{-L(d-c)}\delta_1$. Using the description of U_δ and $\overline{U}_\delta$ in (2.1.3) and (2.1.4) we see that if $\delta' < \delta''$ then $\overline{U}_{\delta'} \subset U_{\delta''}$, since $[c, d] \times \overline{B}(\mathbf{0}, \delta') \subset [c, d] \times B(\mathbf{0}, \delta'')$.

Let $\delta_m = e^{-L(d-c)}\delta_1$. Note that $U = U_{\delta_m}$. Pick $\delta \in \mathbf{D}$. It is clear that δ satisfies parts (a) and (b) of the theorem since $U_\delta \subset U$. It remains to prove (c). Since $\delta < \delta_m$, from the observations in the last paragraph, $\overline{U}_\delta \subset U$. Thus $\boldsymbol{F}$ is defined and continuous on $[c, d] \times \overline{U}_\delta$. Since $[c, d] \times \overline{U}_\delta$ is compact, $\boldsymbol{F}$ is uniformly continuous on it. It follows that $\boldsymbol{F}$ is uniformly continuous on $[c, d] \times U_\delta$. This proves (c) for every $\delta \in \mathbf{D}$. $\square$

References

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