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Lecture 2

Recall that the main interest of this course is an IVP (initial value problem) of the following kind:

(1) $$\dot{\vec{x}} = \vec{v}(t, \vec{x}), \quad \vec{x}(t_0) = \vec{x}_0$$

Here, t varies in an open interval I (called the time space) in $\mathbb{R}$, and $\vec{x}$ varies in an open subset Ω in $\mathbb{R}^n$ (Ω is called the phase space or the state space) and

$$\vec{v} : I \times \Omega \longrightarrow \mathbb{R}^n$$

is a suitable map (usually at least C^1 in $\vec{x}$). Also $t_0 \in I$ and $\vec{x}_0 \in \Omega$ are fixed points called the initial time point and the initial state.

A solution of (1) is a pair $(J, \vec{\varphi})$ where J is an open interval containing t_0, and

$$\vec{\varphi} : J \longrightarrow \Omega$$

is a diff'ble map such that

$$\dot{\vec{\varphi}}(t) = \vec{v}(t, \vec{\varphi}(t)), \quad t \in J, \quad \vec{\varphi}(t_0) = \vec{x}_0.$$

The interval J is called an interval of existence.

Notations: As in Analysis II, $\mathbb{R}^n$ consists of column vectors. We distinguish between $[x_1 \cdots x_n] = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^T$, and $(x_1, \dots, x_n)$.

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= (v_1, \dots, v_n) \qquad = (x_1, \dots, x_n)$$

Suppose $(J, \vec{\varphi})$ is a solution of (1).

Consider $\vec{\psi}$ given by

$$\vec{\psi}(t) = (t, \varphi_1(t), \dots, \varphi_n(t))$$

where $(\varphi_1, \dots, \varphi_n) = \vec{\varphi}$.

Let $\hat{\Omega} = I \times \Omega$, and let

$$\vec{w} : \hat{\Omega} \longrightarrow \mathbb{R}^{n+1}$$

be the map

$$\underset{\substack{\Uparrow \\ \hat{\Omega}}}{(s, \vec{x})} \longmapsto (1, \vec{v}(s, \vec{x})) = (1, v_1(s,x), \dots, v_n(s,x))$$

Consider the IVP

(2) $$\dot{\vec{\xi}} = \vec{w}(\vec{\xi}), \qquad \vec{\xi}(t_0) = \begin{bmatrix} t_0 \\ \vec{x_0} \end{bmatrix}$$

Note that

(a) (2) is autonomous, i.e, the right side does not depend upon I

(b) $\vec{\Psi}$ is a solution of (2).

(c) Given a solution $\vec{\Psi}$ of (2), say $\vec{\Psi} = (\psi_0, \psi_1, \dots, \psi_n)$

then $\psi_0(t) = t$, and if $\phi_i := \psi_i$, then $\vec{\phi} = (\phi_1, \dots, \phi_n)$ is a solution of (1)

Recall that an autonomous IVP is of the form

$$\dot{\vec{x}} = \vec{v}(\vec{x}), \qquad \vec{x}(t_0) = \vec{x_0},$$

where $\vec{v} : \Omega \longrightarrow \mathbb{R}^n$ is a function. For all practical purposes, I can be taken to be $\mathbb{R}$.

$$\vec{\tilde{v}} : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^n, \quad \vec{\tilde{v}}(t, \vec{x}) = \vec{v}(x).$$

Since the study of (1) amounts to the study of (2), we can restrict ourselve to autonomous DE's, i.e. DE's of the form

$$\dot{\vec{x}} = \vec{v}(\vec{x}), \qquad \vec{x}(t_0) = \vec{x_0}.$$

Autonomous eqns when n=1:

Consider the autonomous IVP

(*) $\dot{x} = v(x), \quad x(t_0) = x_0$

where $v : \Omega \longrightarrow \mathbb{R}$ is a $\mathcal{C}^1$ map on an open interval Ω of $\mathbb{R}$, x_0 is a fixed point in Ω, and t_0 a time point.

For any function f, $\mathrm{Dom}(f)$ will denote its domain.

1. Time reversal: Let $\varphi : (a,b) \longrightarrow \Omega$ be a solution of $(*)$. Recall, this implies $t_0 \in (a,b)$. Then the map

$$\varphi^{tr} : (2t_0 - b, 2t_0 - a) \longrightarrow \Omega$$

given by

$$\varphi^{tr}(t) = \varphi(2t_0 - t), \qquad 2t_0 - b < t < 2t_0 - a$$

is a solution of

$$(*)_{tr.} \qquad \dot{x} = -v(x), \quad x(t_0) = x_0$$

The assertion is easy to verify.

The IVP $(*)_{tr}$ is called the time reversal of $(*)$ and the map φ^{tr} is called the time reversal of φ.

2 State reversal: Let $\varphi : (a,b) \longrightarrow \Omega$ be a soln of $(*)$. Set $-\Omega = \{x \in \mathbb{R} \mid -x \in \Omega\}$. Let

$$v^{sr} : -\Omega \longrightarrow \mathbb{R}$$

be the map $v^{sr}(x) = -v(-x)$. Then v^{sr} is $\mathcal{C}^1$ and $v^{sr}(-x_0) = -v(x_0)$. Let

$$\varphi^{sr} : (a,b) \longrightarrow -\Omega$$

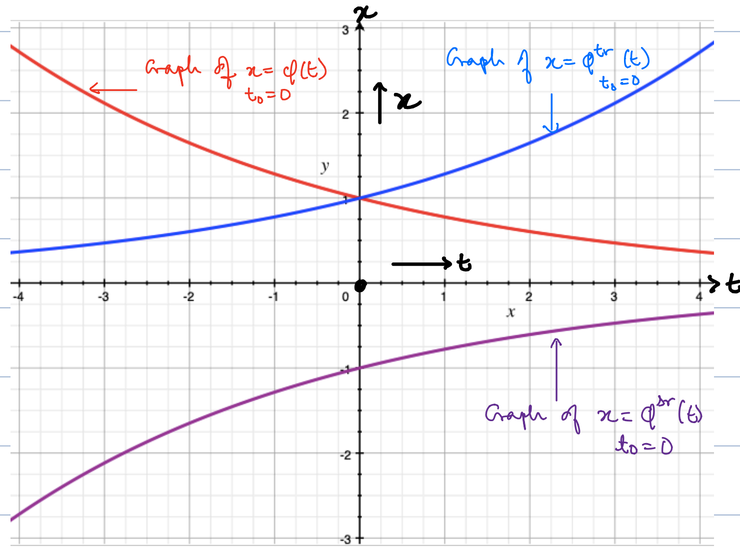
be the map

$$\varphi^{sr}(t) = -\varphi(t), \qquad t \in (a,b).$$

Then φ^{sr} is a soln of the IVP

$(*)_{sr}$ ———— $\dot{x} = v^{sr}(x), \qquad x(t_0) = -x_0.$

Once again, this assertion is easy to verify. The IVP $(*)_{sr}$ is called the <u>state reversal of $(*)$</u> and φ^{sr} the <u>state reversal of φ</u>.



Regular and singular states:

A state (a.k.a. phase) $x \in \Omega$ is said to be <u>regular</u> if $v(x) \neq 0$. Otherwise it is called <u>stationary</u> or <u>singular</u>, i.e. x is stationary (or singular) if $v(x) = 0$.

$$\Omega^{reg} = \{x \in \Omega \mid v(x) \neq 0\} \longleftarrow \text{Open set of } \Omega / \mathbb{R}.$$

$$\Omega^{sing} = \{x \in \Omega \mid v(x) = 0\} \longleftarrow \text{Closed subset of } \Omega$$

$\longrightarrow$ P.T.O.

Suppose $x_0 \in \Omega^{reg}$. Let

$$S_{max} = (x_m, x_M) \subset \Omega^{reg}$$

be the largest interval containing x_0 which lies in Ω^{reg}. In other words S_{max} is the connected component of the open set Ω^{reg} containing x_0.

Since v is nowhere vanishing on S_{max}, it has a constant sign on it. Let

$$\theta : S_{max} = (x_m, x_M) \longrightarrow \mathbb{R}$$

be the function defined by

$$\theta(x) = t_0 + \int_{x_0}^{x} \frac{d\xi}{v(\xi)}, \quad x \in S_{max}.$$

Since $v(\xi)$ has a constant sign for $\xi \in S_{max}$, θ is strictly monotone, and hence one-to-one. Moreover θ is continuous (in fact $\mathcal{C}^2$). Let

$$\theta(S_{max}) = (\omega_-, \omega_+) =: J_{max}.$$

Let $\varphi_{max} : \underset{(\omega_-,\, \omega_+)}{\underset{\parallel}{J_{max}}} \longrightarrow \underset{(x_m,\, x_M)}{\underset{\parallel}{S_{max}}}$ be the inverse function of θ, i.e., $\varphi_{max} = \theta^{-1}$. The inverse function theorem shows that φ_{max} is diff'ble and $\mathcal{C}^2$.

Now

$$\theta(\varphi_{max}(t)) = t.$$

Hence

$\frac{1}{v} \rightarrow$

$$\frac{d\theta}{dx}(\varphi_{max}(t)) \cdot \dot{\varphi}_{max}(t) = 1.$$

Hence $$\frac{\dot{\varphi}_{max}(t)}{v(\varphi_{max}(t))} = 1$$

So $$\dot{\varphi}_{max}(t) = v(\varphi_{max}(t))$$

Also

$$\theta(x_0) = t_0$$

Hence $\varphi_{max}(t_0) = x_0$.

Thus $\varphi_{max} : (\omega_-, \omega_+) = J_{max} \longrightarrow \Omega$

is a solution of (*).