LECTURE 19

Date of the Lecture: March 17, 2021

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \ \|_2$ and we will simply denote it as $\| \ \|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\| \ \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n, \mathbf{K}^n)$.

Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. Basic notations and conventions

1.1. Throughout this lecture we assume that $\boldsymbol{v} \colon \Omega \to \mathbf{R}^n$ is a continuous function which is Lipschitz in $\boldsymbol{x}$ with Lipschitz constant L on a domain[2] Ω contained in $\mathbf{R} \times \mathbf{R}^n$. Our interest is in the behaviour of solutions of the differential equation

$$(\Delta) \qquad \dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$$

as we vary the initial conditions, i.e. the initial time point and the initial state. To that end, for a point $\boldsymbol{\xi} = (\tau, \boldsymbol{a})$ in Ω, the symbol $\boldsymbol{\varphi}_{(\tau,\boldsymbol{a})}$ (or simply $\boldsymbol{\varphi}_{\boldsymbol{\xi}}$) will denote the unique solution to initial value problem $(\Delta)_{(\tau,\, \boldsymbol{a})}$ below.

$$(\Delta)_{(\tau,\boldsymbol{a})} \qquad \dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x}), \quad \boldsymbol{x}(\tau) = \boldsymbol{a}.$$

We might often write $(\Delta)_{\boldsymbol{\xi}}$ for $(\Delta)_{(\tau,\boldsymbol{a})}$. For $\boldsymbol{\xi} = (\tau, \boldsymbol{a}) \in \Omega$, $J(\boldsymbol{\xi}) = J(\tau, \boldsymbol{a})$ will denote the maximal interval of existence for $(\Delta)_{(\tau,\, \boldsymbol{a})}$.

[1] See §§2.1 of Lecture 5 of ANA2.

[2] i.e. a non-empty connected open subset of $\mathbf{R}^n$

2. Estimates

Much of the material in the rest of this lecture is taken from [CL].

2.1. The fundamental estimate. Recall the following definition from HW 4:

Definition 2.1.1. *We say that $\boldsymbol{\varphi}$ is an ϵ*-approximate solution of (Δ) on an interval I if $(t, \boldsymbol{\varphi}(t))$ is in the domain of $\boldsymbol{v}$ for all $t \in I$ and

$$\left\| \dot{\boldsymbol{\varphi}}(t) - \boldsymbol{v}(t, \boldsymbol{\varphi}(t)) \right\| \le \epsilon \qquad (t \in I).$$

Next suppose $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are $\mathscr{C}^1$ functions on an interval I, with $\boldsymbol{\varphi}$ an ϵ_1-approximation of $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$ on I and $\boldsymbol{\psi}$ an ϵ_2-approximation of $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$ on I. Suppose further that at a specified point τ_0 in I we have $\|\boldsymbol{\varphi}(\tau_0) - \boldsymbol{\psi}(\tau_0)\| \le \delta$. From Problem 8 of HW 4 we have we have the following *fundamental estimate*:

$$\left\| \boldsymbol{\varphi}(t) - \boldsymbol{\psi}(t) \right\| \le \delta e^{L|t-\tau_0|} + \frac{\epsilon_1 + \epsilon_2}{L}\left(e^{L|t-\tau_0|} - 1\right) \qquad (t \in I). \tag{2.1.2}$$

The fundamental estimate can be used to give the following estimate.

Lemma 2.1.3. *Suppose $\boldsymbol{v}$ is bounded and $M < \infty$ an upper bound for $\boldsymbol{v}$ and $\boldsymbol{\xi} = (\tau, \boldsymbol{a})$ a point in Ω. Let $[c, d]$ be an interval of existence for $(\Delta)_{\boldsymbol{\xi}}$ such that $[c, d] \times \{\boldsymbol{a}\} \subset \Omega$. Then*

$$\left\| \boldsymbol{\varphi}_{\boldsymbol{\xi}}(t) - \boldsymbol{a} \right\| \le \frac{M}{L}\left(e^{L|t-\tau|} - 1\right) \qquad (t \in [c, d]).$$

Proof. Let $\boldsymbol{\psi}\colon [c, d] \to \mathbf{R}^n$ be the constant map $\boldsymbol{\psi} \equiv \boldsymbol{a}$. Then for $t \in [c, d]$

$$\|\dot{\boldsymbol{\psi}}(t) - \boldsymbol{v}(t, \boldsymbol{\psi}(t))\| = \|\boldsymbol{v}(t, \boldsymbol{\psi}(t))\| \le M.$$

Thus $\boldsymbol{\psi}$ is an M-approximate solution of (Δ). On the other hand $\boldsymbol{\varphi}_{\boldsymbol{\xi}}$ is an exact solution (Δ). By the fundamental estimate, with $\epsilon_1 = M$, $\epsilon_2 = 0$, and $\delta = 0$, we get

$$\|\boldsymbol{\varphi}_{\boldsymbol{\xi}}(t) - \boldsymbol{a}\| = \|\boldsymbol{\varphi}_{\mathbf{0}}(t) - \boldsymbol{\psi}(t)\| \le \tfrac{M}{L}(e^{L|t-\tau|} - 1)$$

for every $t \in [c, d]$. ☐

3. Subsets of Ω

3.1. For the rest of this section, we fix a solution $\boldsymbol{\varphi}\colon [c,\, d] \to \Omega$ of the differential equation (Δ). For each $\delta > 0$, let

$$U_\delta = \left\{(\tau, \boldsymbol{a}) \in \mathbf{R}^{n+1} \mid \tau \in [c, d],\ \|\boldsymbol{a} - \boldsymbol{\varphi}(\tau)\| < \delta\right\}, \tag{3.1.1}$$

Lemma 3.1.2. *There exists $\delta_1 > 0$ such that the closure $\overline{U}_{\delta_1}$ of U_{δ_1} in $\mathbf{R}^n$ is a compact subset of Ω.*

Proof. The map $f\colon [c, d] \times \mathbf{R}^n \to [c, d] \times \mathbf{R}^n$ given by

$$f(t, \boldsymbol{a}) = (t, \boldsymbol{a} + \boldsymbol{\varphi}(t))$$

is a homeomorphism. Indeed, if $\pi_1\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is the projection $(t,\, \boldsymbol{a}) \mapsto t$ and $\pi_2\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ the projection $(t, \boldsymbol{a}) \mapsto \boldsymbol{a}$, then $f = (\pi_1, \pi_2 + \boldsymbol{\varphi} \circ \pi_1)$, and hence is continuous. It has an inverse g given by $g(t, \boldsymbol{\xi}) = (t, \boldsymbol{\xi} - \boldsymbol{\varphi}(t))$, and since $g = (\pi_1, \pi_2 - \boldsymbol{\varphi} \circ \pi_2)$, it too is continuous. Thus f is a homeomorphism. This means $U_\delta = f\big([c, d] \times B(\mathbf{0}, \delta)\big)$ an open subset of $[c, d] \times \mathbf{R}^n$ for every $\delta > 0$. (See Figure 1 below.)

Next let $\Omega' = f^{-1}(\Omega \cap ([c,d] \times \mathbf{R}^n))$. Then Ω' is open in $[c,d] \times \mathbf{R}^n$ and contains $[c,d] \times \{\mathbf{0}\}$. If $d(t, \boldsymbol{a})$ is the distance from $(t, \boldsymbol{a})$ to the closed set $([c,d] \times \mathbf{R}^n) \smallsetminus \Omega'$ of $[c,d] \times \mathbf{R}^n$, then d is continuous on $[c,d] \times \mathbf{R}^n$. Since $K = [c,d] \times \{\mathbf{0}\}$ is compact, the infimum of d on K is a positive number η. Pick $\delta_1 < \eta$. Then $[c,d] \times \overline{B}(\mathbf{0}, \delta_1) \subset \Omega'$. It follows that $\overline{U}_{\delta_1}$ (which equals $f([c,d] \times \overline{B}(\mathbf{0}, \delta_1))$) is compact and contained in $\Omega \cap ([c,d] \times \mathbf{R}^n)$. □

3.1.3. We will show that there is a δ with $0 < \delta < \delta_1$ such that $[c,d]$ is an interval of existence for $(\Delta)_{\boldsymbol{\xi}}$ for every $\boldsymbol{\xi} = (\tau, \boldsymbol{a})$ in U_δ. Note that this is not the same as the assertion in 1.1.2 of Lecture 6 though it is closely related to that.

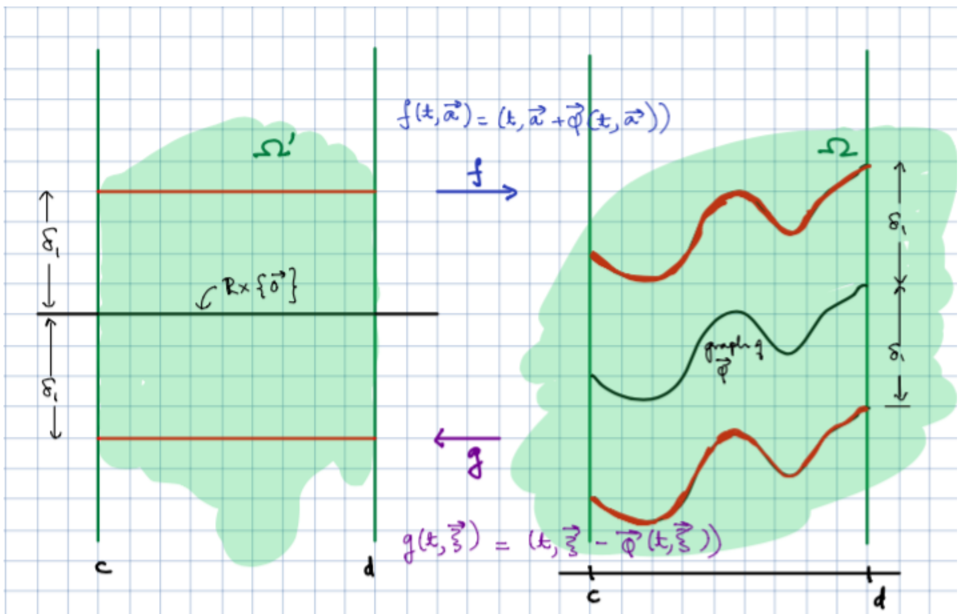

FIGURE 1. U_{δ_1} is the region between the two brown curves in the figure on the right

The next lecture we will prove the following:

Theorem 3.1.4. *With the above notations, there exists a* $\delta > 0$*, such that*

(a) $U_\delta \subset \Omega$.
(b) *For every* $\boldsymbol{\xi} = (\tau, \boldsymbol{a}) \in U_\delta$ *the solution* $\boldsymbol{\varphi}_{\boldsymbol{\xi}}$ *to* $(\Delta)_{\boldsymbol{\xi}}$ *exists on* $[c,d]$.
(c) *The map* $(t, \tau, \boldsymbol{a}) \mapsto \boldsymbol{\varphi}_{(\tau, \boldsymbol{a})}(t)$ *is uniformly continuous on* $V = [c,d] \times U_\delta$.

Proof. See the next lecture.

References

[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, Third Edition, translated by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.

[CL] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.