LECTURES 17 AND 18

Dates of the Lectures: March 8 and 10, 2021

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \ \|_2$ and we will simply denote it as $\| \ \|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\| \ \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n, \mathbf{K}^n)$.

Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. Compact Manifolds

1.1. **Intervals of existence on compact manifolds.** Recall from the remark in 1.1.2 of Lecture 6 that if Ω is an open subset of $\mathbf{R}^{n+1}$, $\boldsymbol{v}\colon \Omega \to \mathbf{R}^n$ a locally Lipschitz map, then for every point $(\tau, \boldsymbol{a})$ in Ω we have an open neighbourhood $W_{(\tau,\, \boldsymbol{a})}$ in Ω of $(\tau, \boldsymbol{a})$ and a positive number $b = b_{(\tau,\, \boldsymbol{a})}$ such that $(\tau - b, \tau + b)$ is an interval of existence for the initial value problem $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$, $\boldsymbol{x}(\theta) = \boldsymbol{y}$ for every $(\theta, \boldsymbol{y})$ in $W_{(\tau,\, \boldsymbol{a})}$. Since this is a local statement, it clearly holds for manifolds. We state this (and more) formally below.

[1] See §§2.1 of Lecture 5 of ANA2.

Theorem 1.1.1. *Let M be a manifold of dimension n, Ω an open subset of $\mathbf{R}\times M$, and $v\colon \Omega \to T(M)$ a $\mathscr{C}^1$ map such that the diagram*

$$\begin{tikzcd} & T(M) \arrow[d, "\varpi"] \\ \Omega \arrow[ur, "v"] \arrow[r] & M \end{tikzcd}$$

commutes where the horizontal arrow at the bottom is the second projection.

(a) *For every point $(\tau, a) \in \Omega \subset \mathbf{R}\times M$, we have an open neighbourhood $W_{(\tau,a)}$ of (τ, a) in Ω and a positive real number $b = b_{(\tau,a)}$ such that for each point (θ, y) in $W_{(\tau,a)}$ the interval $(\tau - b, \tau + b)$ is an interval of existence for the initial value problem $\dot{x} = v(t, x)$, $x(\theta) = y$.*

(b) *Let K be a compact subset of Ω. There exists a positive real number b_K (depending only on K and $\boldsymbol{v}$) such that for every $(\tau, a) \in K$, the open interval $(\tau - b_K, \tau + b_K)$ is an interval of existence for the initial value problem $\dot{x} = \boldsymbol{v}(t, x)$, $x(\tau) = a$.*

(c) *Let $(t_0, a_0) \in \Omega$ and let $\varphi_0\colon J \to M$ be the solution of the initial value problem $\dot{x} = v(t, x)$, $x(t_0) = a_0$ with $J = (\omega_-, \omega_+)$ the maximal interval of existence for the IVP. Then the variable point $(t, \varphi_0(t))$ leaves every compact set K in Ω as $t \uparrow \omega_+$ and $t \downarrow \omega_-$.*

Proof. Part (a) is a local statement, and hence we may assume M is an open subset of $\mathbf{R}^n$. Then part (a) follows from 1.1.2 of Lecture 6. We are of course using the fact that for any co-ordinate chart (U, ψ) of M, the corresponding DE on $\psi(U) \subset \mathbf{R}^n$ is of the form $\dot{\boldsymbol{y}} = \boldsymbol{w}(t, \boldsymbol{y})$ with $\boldsymbol{w}$ a $\mathscr{C}^1$ map, whence locally Lipschitz.

To prove part (b), we use part (a) to cover K by $W_{(t,z)}$ as (t, z) varies over K, and extract a finite subcover $\{W_{(t_i,z_i)}\}_{i=1}^k$. Let $b_K = \min\{b_{(t_i,z_i)} \mid i = 1, \ldots, k\}$. It is clear that $(\tau - b_K, \tau + b_K)$ is an interval of existence for each of the initial value problems $\dot{x} = v(t, x)$, $x(\tau) = a$ as (τ, a) varies in K.

The proof of (c) is identical to the one given in the last paragraph of the proof of Theorem 1.1.1 of Lecture 6. In other words, if $\tau \in (\omega_+ - b_K, \omega_+)$ then $(\tau, \varphi_0(\tau)) \notin K$, and likewise, when $\tau \in (\omega_-, \omega_- + b_K)$ then $(\tau, \varphi_0(\tau)) \notin K$. If either of the intervals $(\omega_+ - b_K, \omega_+)$ or $(\omega_-, \omega_- + b_K)$ is empty (i.e. if either $\omega_+ = \infty$ or $\omega_- = -\infty$) then again the argument given in *loc.cit.* stands. For example, if $\omega_+ = \infty$, then set τ_M equal to the maximum of τ such that $(\tau, \boldsymbol{\varphi}_0(\tau)) \in K$. Then $\omega_- < \tau_M < \infty$ since K is compact, and $(\tau, \boldsymbol{\varphi}_0(\tau)) \notin K$ for $\tau > \tau_M$. A similar argument shows that when $\omega_- = -\infty$, there exists $\tau_m \in (-\infty, \omega_+)$ such that $(\tau, \boldsymbol{\varphi}_0(\tau)) \notin K$ for $\tau < \tau_m$. □

The following corollary will be the guiding light for what we do in the remaining part of this lecture and for most of the next lecture.

Corollary 1.1.2. *Let M be a compact n-dimensional manifold and $v\colon M \to T(M)$ a $\mathscr{C}^1$ vector field on it. Let $a \in M$. Then for each $p \in M$, $\mathbf{R}$ is an interval of existence of the autonomous initial value problem $\dot{x} = v(x)$, $v(x) = p$.*

Proof. Let $\Omega = \mathbf{R}\times M$ and set $K = \{0\} \times M$. Then K is a compact subset of Ω. By part (b) of the theorem, we have a positive real number b_K such that for every $p \in M$, the interval $(-b_K, b_K)$ is an interval of existence for $\dot{x} = v(x)$, $x(0) = p$. By Problem 7 of the mid-term exam (whose solutions are here) we are done (take $\varepsilon = b_K$). □.

2. **The logistic equation**

We will look at the autonomous differential equation, the so called *logistic equation*, which is

$$\dot{y} = y(1-y) \tag{1}$$

The equilibrium solutions are $y \equiv 0$ and $y \equiv 1$. Let us take our initial time point to be $t_0 = 0$.

2.1. If we consider the IVP associated to (1), with initial condition$y(0) = y_0$, where y_0 is not an equilibrium state, i.e., $y_0 \notin \{0,1\}$, then from general theory we know that a solution $\varphi(t)$ cannot lie in $\{0,1\}$ for any time point t in its domain. Indeed, if $\varphi(t_1) \in \{0,1\}$ for some $t_1 \in \mathbf{R}$, then by the uniqueness of the solution (using the fact that $y \mapsto y(1-y)$ is a $\mathscr{C}^1$ function) to the IVP, $y' = y(1-y)$, $y(t_1) = y_1$, we see that $\varphi(t) \in \{0,1\}$ for all $t \in \mathbf{R}$, contradicting the fact that $\varphi(0) = y_0 \notin \{0,1\}$.

We assume now that $y_0 \notin \{0,\, 1\}$. The DE (1) is solved formally by noting that the equation is equivalent to $\frac{dy}{y(1-y)} = dt$, and that

$$\frac{1}{y(1-y)} = \left\{\frac{1}{y} - \frac{1}{y-1}\right\}.$$

From this one sees that

$$\begin{aligned} t &= \ln\left|\frac{y}{y-1}\right| - \ln\left|\frac{y_0}{y_0-1}\right| \\ &= \ln\left|\frac{y}{y_0}\frac{y_0-1}{y-1}\right| \end{aligned}$$

Let C be one of the three connected components of $\mathbf{R} \smallsetminus \{0,\, 1\}$, namely $(-\infty, 0)$, $(0,1)$, and $(1,\infty)$. From our arguments above, if $y_0 \in C$, then $y(t) \in C$ for all t in the maximal interval of existence of $\dot{y} = y(1-y)$ with $y(0) = y_0$. Here of course $y(t)$ is the solution of this IVP. It follows quite easily from this that $\frac{y}{y_0}$ is positive, as is $\frac{y_0-1}{y-1}$. Thus

$$t = \ln\left\{\frac{y_0-1}{y_0}\frac{y}{y-1}\right\}.$$

Inverting we get

$$y(t) = \frac{y_0 e^t}{y_0 e^t - y_0 + 1}, \qquad (t \in J(y_0)) \tag{2.1.1}$$

where $J(y_0)$ is the maximal interval of existence for solutions of (1) satsfying $y(0) = y_0$. We point out the formula (2.1.1) is true even when $y_0 \in \{0,1\}$, as can be checked by directly plugging in the values 0 and 1 for y_0 in the formula.

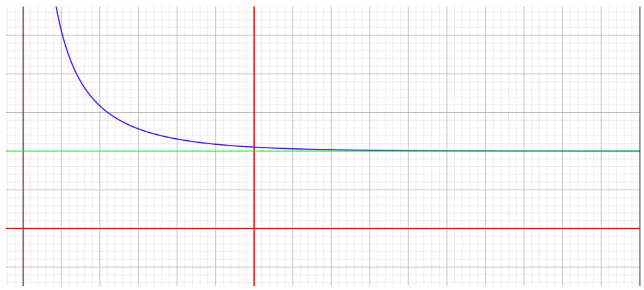
It is easy to see, by direct calculations, that the RHS of (2.1.1) gives a solution for our DE at all time intervals on which the denominator never vanishes.

If the connected component C of $\mathbf{R} \smallsetminus \{0,\, 1\}$ we are working over is $(0,1)$ we note that the denominator on the RHS of (2.1.1) is never zero. Indeed the denominator $y_0e^t - y_0 + 1$ is an increasing function of t and its limit as $t \downarrow -\infty$ is $1 - y_0$, whence $y_0e^t - y_0 + 1 > 1 - y_0 > 0$ for all $t \in \mathbf{R}$. Thus $J(y_0) = \mathbf{R}$ in this case.

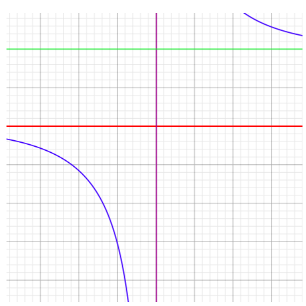
If C is one of the other two components, i.e. one of $(-\infty, 0)$ or $(1,\infty)$, then the equation $y_0e^t - y_0 + 1 = 0$ has a solution, namely $t = t_\infty$ where $t_\infty = \ln\{\frac{y_0-1}{y_0}\}$.

If $y_0 > 1$, then it is easy to see that $0 < \frac{y_0 - 1}{y_0} < 1$, and if $y_0 < 0$ then $\frac{y_0-1}{y_0} > 1$. This means $t_\infty < 0$ when $y_0 > 1$ and $t_\infty > 0$ when $y_0 < 0$. It follows that $J(y_0) = (t_\infty, \infty)$ if $y_0 > 1$ and $J(y_0) = (\infty, t_\infty)$ when $y_0 < 0$.

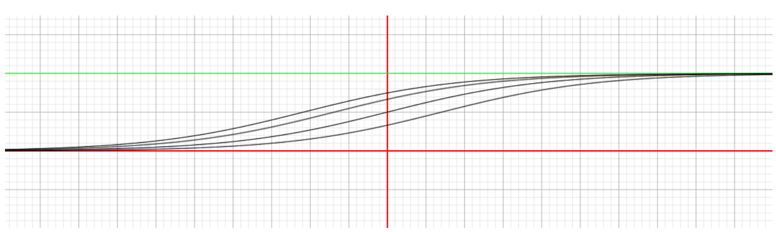
2.2. **The case $y_0 > 1$.** Here is the graph of $y(t)$ when $y_0 = 20/19$.



The purple vertical line on the left is at $t = -\ln 20$. The red lines are the t and y axes (the t-axis is horizontal) and the green line is the line $y = 1$. The graph of y is blue coloured. Note that the solution blows up to infinity as $t \to t_\infty = -\ln 20$ from the right, the solution approaches the equilibrium $y = 1$ as $t \to \infty$. To left of t_∞, the graph of the "solution" looks like the graph below, but, one cannot reach it from our initial phase (travelling backwards in time) of $y(0) = y_0 = 20/19$. So we do not regard it as a solution of our IVP, even though it is a solution to the DE.

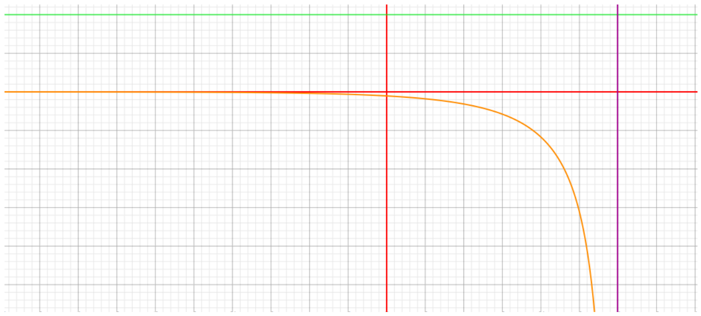


2.3. **The case $0 < y_0 < 1$.** Here are graphs of the above with $y_0 = 1/3$, $1/2$, $2/3$, and $3/4$.

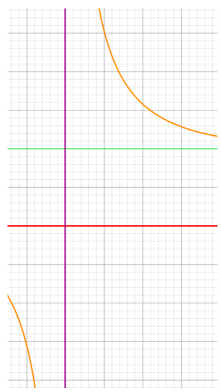


As before, the red lines are the axes, and the green line is the line $y = 1$.

2.4. **The case $y_0 < 0$.** Cutting to the chase, for $y_0 = -1/19$, the graph (the orange curve) is as follows (the purple line being $t = t_\infty$, i.e., $t = \ln 20$.)



As before, there is a formal solution beyond $t = \ln 20$ but that cannot be considered as part of the solution of our IVP with $y_0 = -1/19$. For completeness, here is how that part looks:



The logistic equation extended to the circle

The 3D graphics below might help you understand the role of compactness in autonomous systems. And help you appreciate manifolds (versus open sets in $\mathbf{R}^n$).

2.5. **The circle as a manifold.** We can regard the standard unit circle $\mathbf{P}$ in $\mathbf{R}^2$ as differential manifold in many ways. To begin with, it is a level curve of $f(x,y) = x^2+y^2$, and on the unit circle, it is clear that the rank of $f'(x,y) = (2x, 2y)$ is 1. Hence by our now familiar technique (using the implicit function theorem), the unit circle is indeed a manifold. Once can also view it as the manifold obtained by gluing two copies of $\mathbf{R}$ along $\mathbf{R} \smallsetminus \{0\}$ via the diffeomorphism $y \mapsto y^{-1}$ on $\mathbf{R} \smallsetminus \{0\}$. The descriptions are equivalent as you can check by yourself. In this section we will use the second description.

In greater detail, for us $\mathbf{P}$ has an open cover $\mathscr{U} = \{U, V\}$, where $U = \mathbf{P} \smallsetminus \{N\}$, $V = \mathbf{P} \smallsetminus \{S\}$. Here N is the "north pole" $(0,1)$ and S is the "south pole" $(0,-1)$. We have homeomorphisms $\psi_U : U \to \mathbf{R}$ and $\psi_V : V \to \mathbf{R}$ and the transition function is the diffeomorphism $\mathbf{R} \smallsetminus \{0\} \to \mathbf{R} \smallsetminus \{0\}$ given by $y \mapsto z = y^{-1}$.

2.6. **The logistic equation on the unit circle.** Let $w_0 \colon \mathbf{R} \to \mathbf{R}$ be the map $w_0(y) = y(1-y)$. This is of course $\mathscr{C}^1$. Regarding w_0 as a vector field on $\mathbf{R}$, v should be interpreted as the derivation $y(1-y)\frac{\mathrm{d}}{\mathrm{d}y}$. Under the transition map

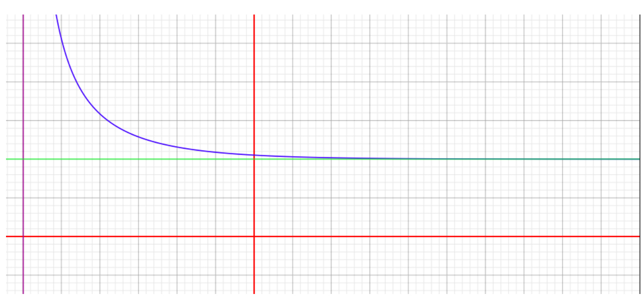
$y \mapsto z = y^{-1}$ this derivation transforms as follows:

$$\begin{aligned} y(1-y)\frac{\mathrm{d}}{\mathrm{d}y} &= y(1-y)\frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}z} \\ &= -\frac{1}{y^2}y(1-y)\frac{\mathrm{d}}{\mathrm{d}z} \\ &= z^2 y(y-1)\frac{\mathrm{d}}{\mathrm{d}z} \\ &= (1-z)\frac{\mathrm{d}}{\mathrm{d}z}. \end{aligned}$$

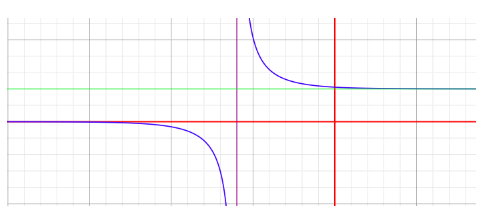
Thus, the restriction of the differential equation $\dot{y} = y(1-y)$ to $\mathbf{R} \setminus \{0\}$ transforms under the transition function $\psi_V \circ \psi_U^{-1} \colon \mathbf{R} \setminus \{0\} \to \mathbf{R} \setminus \{0\}$ to the differential equation $\dot{z} = 1 - z$. This can also be seen in the following way (the naive high-school calculus substitution method): $\frac{\mathrm{d}y}{\mathrm{d}t} = y(1-y)$ is (under the substitution $y = z^{-1}$) the same as the equation $-z^{-2}\frac{\mathrm{d}z}{\mathrm{d}t} = y(1-y)$, and this simplifies to $\frac{\mathrm{d}z}{\mathrm{d}t} = z^2 y(y-1)$, i.e. $\frac{\mathrm{d}z}{\mathrm{d}t} = 1 - z$. Let us denote the vector field $(1-z)\frac{\mathrm{d}}{\mathrm{d}z}$ on $\mathbf{R}$ by w_1.

The upshot is this: We have a vector field v_0 on U and v_1 on V obtained by transporting the vector fields w_0 and w_1 to U and V respectively via ψ_U^{-1} and ψ_V^{-1} and v_0 and v_1 glue on $\mathbf{P}$ to give us a $\mathscr{C}^1$ vector field v on $\mathbf{P}$. Now $\mathbf{P}$ is compact. So we know, via Corollary 1.1.2, that the intervals of existence associated with the autonomous DE $\dot{x} = v(x)$ are $\mathbf{R}$. Here are the graphics illustrating this.

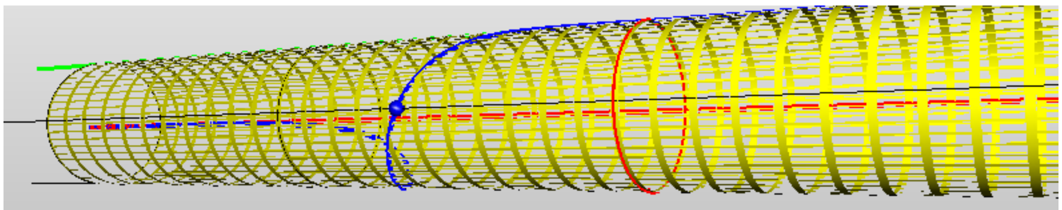
The case $y_0 > 1$. Here is the solution curve of our original DE when $y_0 = 20/19$ (the blue curve):



The purple vertical line on the left is at $t = -\ln 20$. The red lines are the t and y axes (the t-axis is horizontal) and the green line is the line $y = 1$. The vertical asymptote (the purple line) that you see is $t = -\ln 20$. The graph $y(t) = \frac{20e^t}{20e^t - 1}$ "continues" to the left of the the break, and the full graph is:

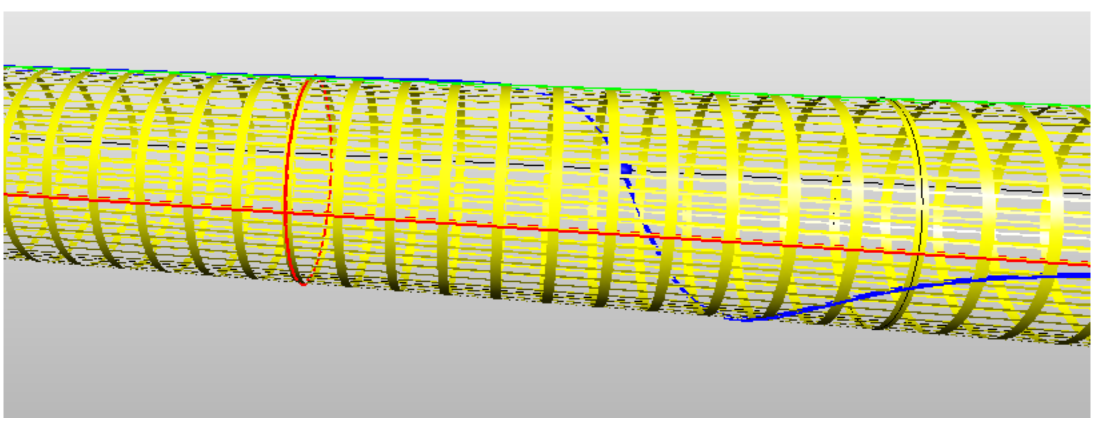


The fact that the process with initial state $y_0 = 20/19$ cannot be continued backwards is not unrelated to the fact that $[1, \infty)$ is non-compact. If we replace the phase space with the circle $\mathbf{P}$ the solution makes sense everywhere. The extended phase space $\mathbf{P} \times \mathbf{R}$ is a cylinder, and the integral curve is a curve on the cylinder. Here are three views of this integral curve on the extended phase space:

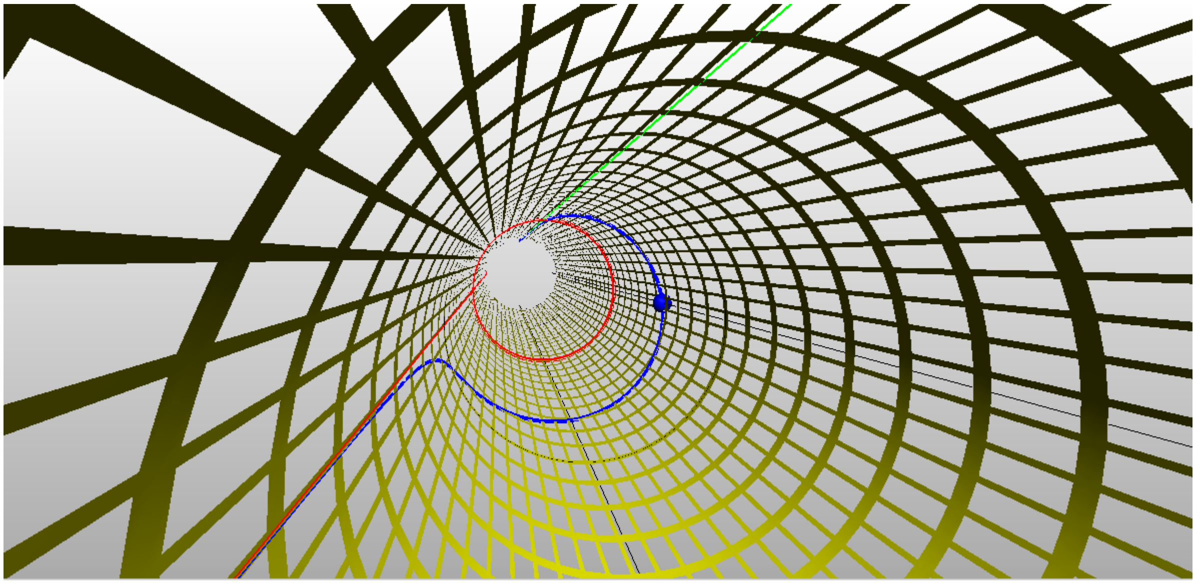


The blue dot is the point on the extended phase space corresponding to $(-\ln 20, \infty)$, ∞ a point on $\mathbf{P}$. The colour code is as before. The old y-axis is now the red circle, and the old t-axis is the red line on the far side of the cylinder. The green line corresponds to the old equilibrium integral curve $y \equiv 1$.

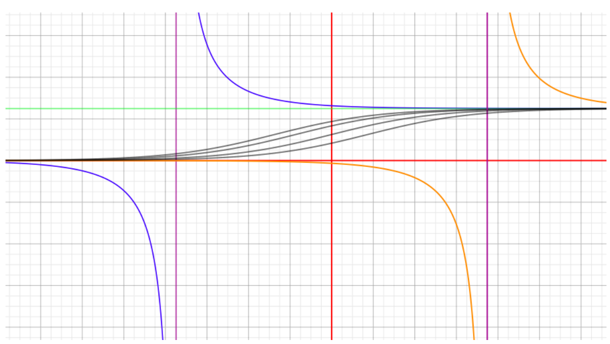
Here is a second view. The postive direction on the t-axis is to the left, because of the way we are viewing the cylinder, namely from the outside.



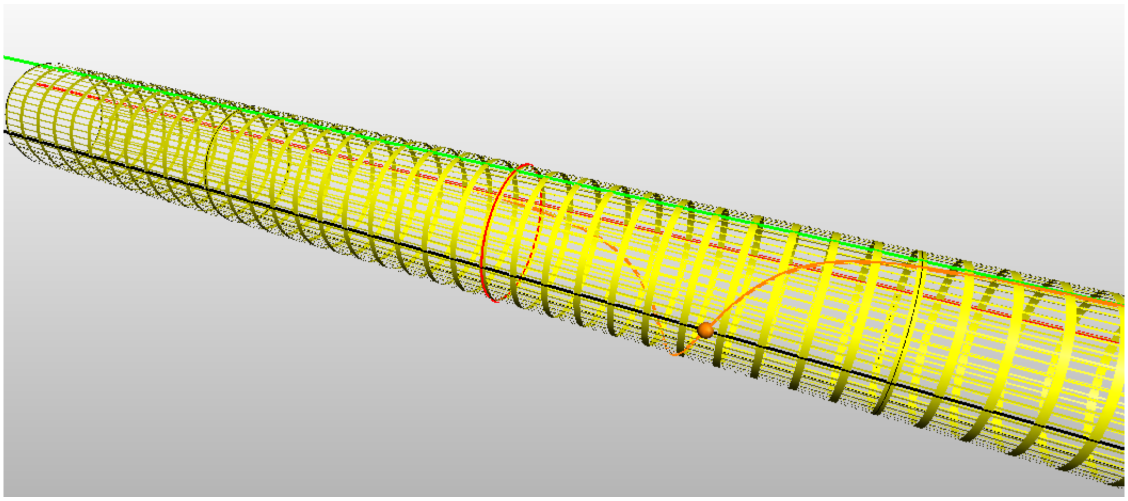
An inside view of the cylinder may be interesting. The blue dot is once again the point at which the integral curve passes through “infinity”.



We had similar graphs for $y_0 \in (0,1)$ and $y_0 < 0$. The graphs we had were $y_0 = 1/3, 1/2, 2/3, 3/4$ (for $y_0 \in (0,1)$ as well as $y_0 = -1/19$, Here are the graphs in one place. The black ones are for the case $y_0 \in (0,1)$ and the orange one (along with its vertical asymptote at $t = \ln 20$), the case where $y_0 = -1/19$.

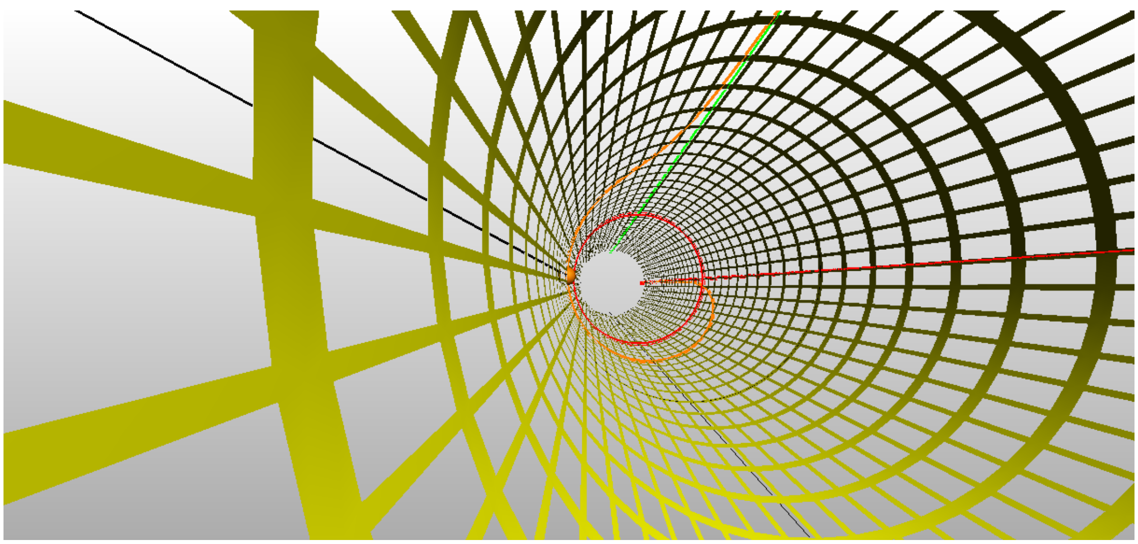


Here are the 3-D graphs when the phase space is compactified to **P**. First we consider the $y_0 < 0$ case. Specifically, the case $y_0 = -1/19$.

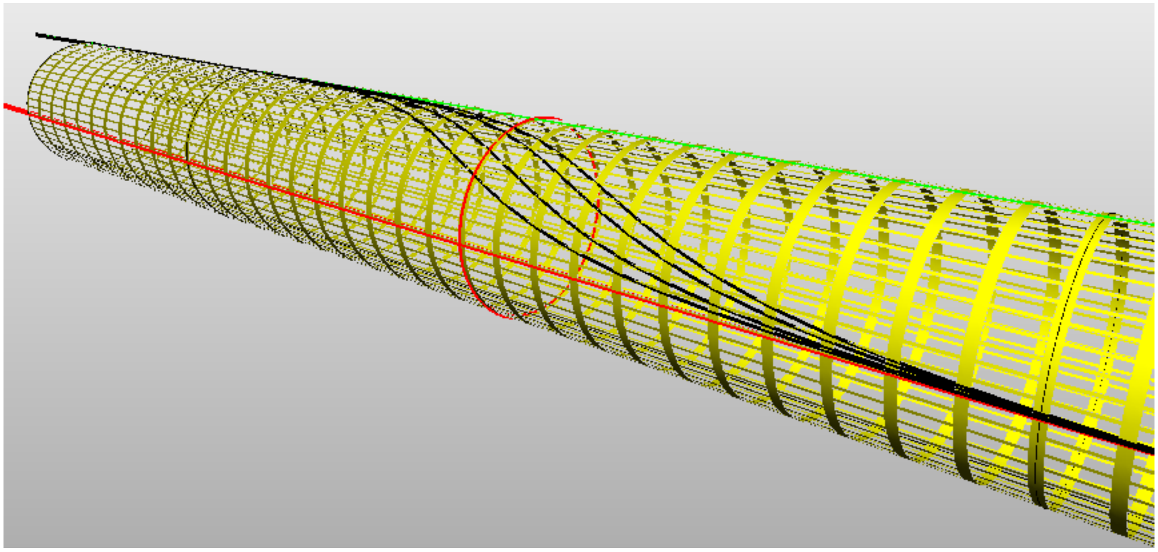


The thick black line is $\mathbf{R} \times \{\infty\}$ and the point where the integral curve crosses it is highlighted as an orange dot. The green line, the red line, and the red circle are as before.

Here is the "inside view" (literally). The line $\mathbf{R} \times \{\infty\}$ is the black line on the left. You can see the orange dot on it, where the integral curve crosses it.

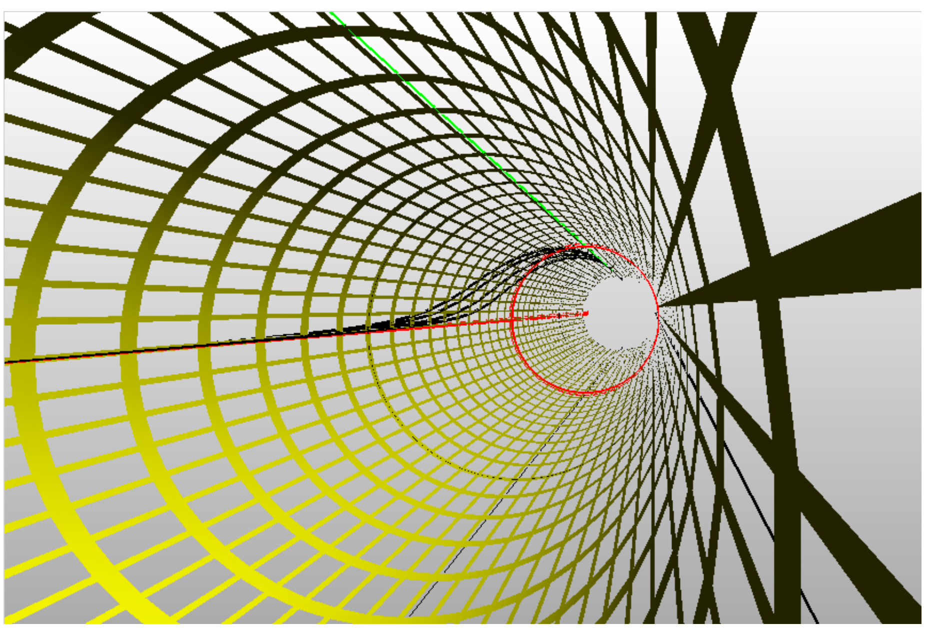


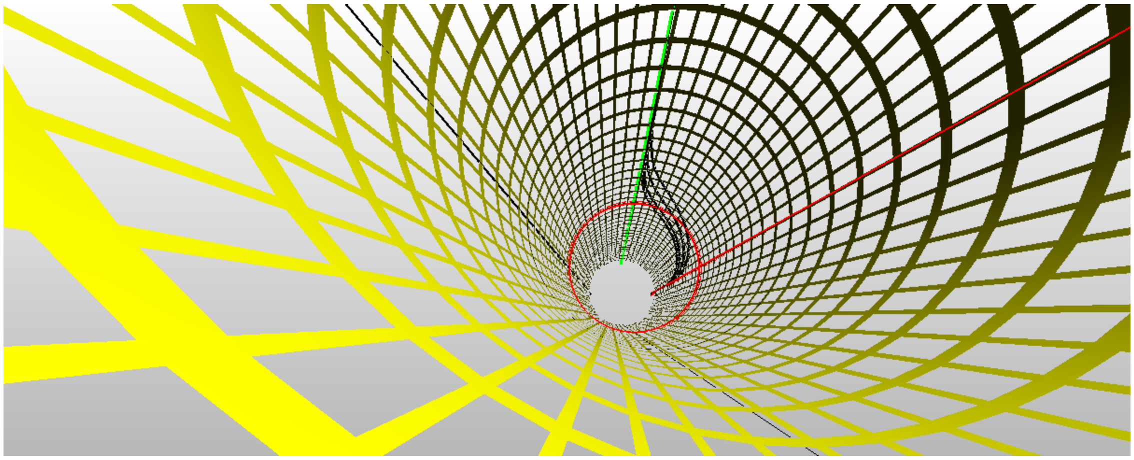
Finally, here are the "black curves", namely the curves occurring when y_0 lies in the interval $(0, 1)$. In our specific examples, the initial phases are $1/3$, $1/2$, $2/3$, and $3/4$. In these cases, the integral curves never touch $\mathbf{R} \times \{\infty\}$.



Note that because of the way $\mathbf{R} \cup \{\infty\} = \mathbf{P}$ has been folded into a circle, the t-axis (the red line) is such that t increases as we move left. The more traditional view of the positive t-axis would would be if we turned the picture around. Or viewed the cylinder from inside (but even here, it depends on which end we peer from).

Here are two inside views, looked at from different ends.





In the above picture, the axis $\mathbf{R} \times \{\infty\}$ can be seen on the left as the black line.

3. Change of coordinates

3.1. **Push-forwards of vector fields.** We have mentioned the obvious fact that under a diffeomorphism, differential equations transform to equivalent differential equations. In fact we just computed the transform of $\dot{y} = y(1-y)$ on $\mathbf{R} \smallsetminus \{0\}$ under the diffeomorphism $y \mapsto y^{-1}$. We now do it a little more systematically.

Let Ω be open in $\mathbf{R}^n$ and assume $\Omega = I \times V$ where V is open in $\mathbf{R}^n$. Let

$$\boldsymbol{F}\colon V \xrightarrow{\sim} W$$

be a diffeomorphism (at least $\mathscr{C}^2$) where W is open in $\mathbf{R}^n$. Let

$$\boldsymbol{G} = \boldsymbol{F}^{-1}$$

Let $\Omega' = I \times W$. Suppose $\boldsymbol{v}\colon \Omega \to \mathbf{R}^n$ is $\mathscr{C}^1$. Let $\boldsymbol{w}\colon \Omega' \to \mathbf{R}^n$ be the map given by

$$\boldsymbol{w}(t, \boldsymbol{y}) = \boldsymbol{F}'(\boldsymbol{G}(\boldsymbol{y}))\boldsymbol{v}(t, \boldsymbol{G}(\boldsymbol{y})). \tag{3.1.1}$$

Note that $\boldsymbol{w}$ is $\mathscr{C}^1$ since $\boldsymbol{F}$ is $\mathscr{C}^2$, whence $\boldsymbol{F}'$ is $\mathscr{C}^1$.

Proposition 3.1.2. *A map $\boldsymbol{\varphi}\colon I \to V$ on an open interval I is a solution of the DE $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$ if and only if $\boldsymbol{\psi} := \boldsymbol{F} \circ \boldsymbol{\varphi}$ is a solution to $\dot{\boldsymbol{y}} = \boldsymbol{w}(t, \boldsymbol{y})$.*

Proof. Since $(\boldsymbol{F}')^{-1}(\boldsymbol{x}) = \boldsymbol{G}'(\boldsymbol{F}(\boldsymbol{x}))$, it is clear that $\boldsymbol{v}(t, \boldsymbol{x}) = \boldsymbol{G}'(\boldsymbol{F}(\boldsymbol{x}))\boldsymbol{w}(t, \boldsymbol{F}(\boldsymbol{x}))$, establishing a symmetry between $\boldsymbol{v}$ and $\boldsymbol{w}$. We therefore only have to prove one direction of the proposition. Let $\boldsymbol{\psi} = \boldsymbol{F} \circ \boldsymbol{\varphi}$ where $\boldsymbol{\varphi}\colon I \to V$ is a solution to $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$ on some open interval I. We have

$$\begin{aligned}
\dot{\boldsymbol{\psi}}(t) &= \boldsymbol{F}'(\boldsymbol{\varphi}(t))\dot{\boldsymbol{\varphi}}(t) \\
&= \boldsymbol{F}'(\boldsymbol{\varphi}(t))\boldsymbol{v}(t, \boldsymbol{\varphi}(t)) \\
&= \boldsymbol{F}'(\boldsymbol{G}(\boldsymbol{\psi}(t)))\boldsymbol{v}(t, \boldsymbol{G}(\boldsymbol{\psi}(t))) \\
&= \boldsymbol{w}(t, \boldsymbol{\psi}(t)).
\end{aligned}$$

Thus $\boldsymbol{\psi}$ is a solution of $\dot{\boldsymbol{y}} = \boldsymbol{w}(t, \boldsymbol{y})$. $\square$

3.1.3. In §§ 2.3 of the notes on derivations, the transformation of vector fields under a change of coordinates is explored. If we regard $\boldsymbol{v}$ as a map $I \times V \to T(V)$ such that $\varpi \circ \boldsymbol{v}$ is the second projection on $I \times V$, and similarly regard $\boldsymbol{w}$ as a map $I \times W \to T(W)$ with $\varpi \circ \boldsymbol{w}$ the second projection, then, in the notation of *loc.cit.*, we have $\boldsymbol{w} = \boldsymbol{F}_* \circ \boldsymbol{v}$. We will use this notation even when regard $\boldsymbol{v}$ and $\boldsymbol{w}$ simply as maps from Ω and Ω' to $\mathbf{R}^n$. The "vector field" $\boldsymbol{w}$ is called the *push-forward* of $\boldsymbol{v}$ under the diffeomorphism $\boldsymbol{F}$.

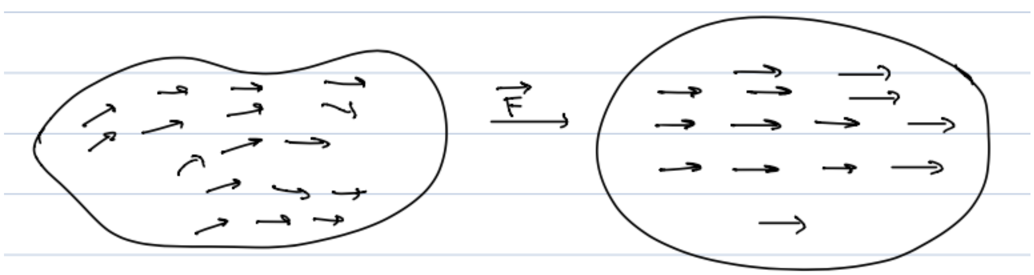
3.2. **The rectification theorem.** The following very important theorem is called the *rectification theorem* or the *flow box* theorem. In some sense, Isaac Barrow, Newton's teacher and the discoverer of the fundamental theorem of Calculus, discovered the rectification theorem, for, substitution as a way of getting anti-derivatives is really rectification in dimension one. We will prove the rectification theorem later in the course.

Theorem 3.2.1. *Let Ω be a domain in $\mathbf{R}^n$, $\boldsymbol{v}$ a $\mathscr{C}^1$ vector field on Ω and Ω^{reg} the subset of Ω on which $\boldsymbol{v}$ is non-zerp. Let $\boldsymbol{x}_0 \in \Omega^{\mathrm{reg}}$. Then there is a neighbourhood U of $\boldsymbol{x}_0$ in Ω^{reg}, a neighbourhood V of $\mathbf{0} \in \mathbf{R}^n$, and a diffeomorphism $\boldsymbol{F}\colon U \xrightarrow{\sim} V$ such that*

$$\boldsymbol{F}(\boldsymbol{x}_0) = \mathbf{0}$$

and

$$\boldsymbol{F}_* \boldsymbol{v} = \boldsymbol{e}_1.$$



References

[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, translated by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.