LECTURE 16

Date of Lecture: March 3, 2021

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \ldots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \, \|_2$ and we will simply denote it as $\| \, \|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\| \, \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n, \mathbf{K}^n)$.

☡ Note that $(x_1, \ldots, x_n) \neq [x_1 \ldots x_n]$. Each side is the transpose of the other.

1. First integrals continued

See also [A1, Chap. 2, § 11] for this material.

1.1. Integral hypersurfaces.

We are interested in solving

$$\dot{x} = v(x)$$

on a manifold M, where $v \colon M \to T(M)$ is a $\mathscr{C}^1$ vector field, using first integrals.

Since solutions are obtained first locally and then by glueing, we can work locally. We will therefore (temporarily, until further notice) assume M is an open subset of $\mathbf{R}^n$. In this case, vector notations make sense, and we can write

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}). \tag{1.1.1}$$

Let the components of $\boldsymbol{v}$ be v_i, $i = 1, \ldots, n$, i.e. $\boldsymbol{v} = (v_1, \ldots, v_n)$, and let $g \colon M \to \mathbf{R}$ be a first integral for $\boldsymbol{v}$ so that $\sum_{i=1}^n v_i \frac{\partial g}{\partial x_i} \equiv 0$ on M.

[1] See §§2.1 of Lecture 5 of ANA2.

If I is an open interval of existence for (1.1.1) and $\boldsymbol{\varphi}\colon I \to M$ a solution of (1.1.1), then we claim that $g \circ \boldsymbol{\varphi}\colon I \to \mathbf{R}$ is constant. Indeed, by the chain rule (with $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{(g\circ\boldsymbol{\varphi})(t)\Big\} &= \sum_{i=1}^{n}\Big(\frac{\partial g}{\partial x_i}(\boldsymbol{\varphi}(t))\Big)\dot{\varphi}_i(t)\\ &= \sum_{i=1}^{n}\Big(\frac{\partial g}{\partial x_i}(\boldsymbol{\varphi}(t))\Big)v_i(\boldsymbol{\varphi}(t)) \qquad \text{(for } \boldsymbol{\varphi} \text{ is a solution of (1.1.1))}\\ &= \boldsymbol{v}(\boldsymbol{\varphi}(t))(g)\\ &= 0\end{aligned}$$

since g is a first integral for $\boldsymbol{v}$.

Now suppose $c \in g(M)$ so that $g^{-1}(c)$ is non-empty. Consider the level hypersurface

$$S = g^{-1}(c).$$

Such S are sometimes called *integral hypersurfaces* for $\boldsymbol{v}$. Fix $\boldsymbol{p}_0 \in S$. Let $t_0 \in \mathbf{R}$, and suppose $\boldsymbol{\varphi}\colon I \to M$ is a solution of

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x}(t_0) = \boldsymbol{p}_0$$

where I is an open interval of existence for the above IVP. Now $g\circ\boldsymbol{\varphi}$ is a constant as we just saw, whence $g(\boldsymbol{\varphi}(t)) = g(\boldsymbol{\varphi}(t_0)) = g(\boldsymbol{p}_0) = c$ for all $t \in I$. Thus $\boldsymbol{\varphi}(t) \in S$ for all $t \in I$. The conclusion is:

Lemma 1.1.2. *Let $\boldsymbol{\varphi}\colon I \to M$ be a solution of (1.1.1) and suppose for some time point t_0 in I, $\boldsymbol{\varphi}(t_0) \in S$. Then $\boldsymbol{\varphi}(t) \in S$ for all $t \in I$.*

1.2. **Functionally independent first integrals.** Next suppose we have $n-1$ first integrals $f_1, \ldots, f_{n-1}$ for $\boldsymbol{v}$ and write $\boldsymbol{f} = (f_1, \ldots, f_{n-1})\colon M \to \mathbf{R}^{n-1}$ for the resulting map. We say $f_1, \ldots, f_{n-1}$ are *functionally independent* on a subset X of M if

$$\operatorname{rank}(\boldsymbol{f}'(\boldsymbol{p})) = n-1, \qquad \boldsymbol{p} \in X.$$

Theorem 1.2.1. *Let M, $\boldsymbol{v}$, $\boldsymbol{f} = (f_1, \ldots, f_{n-1})$ be as above. Suppose $\boldsymbol{v}$ has no singular points on M (i.e. $\boldsymbol{v}(\boldsymbol{x}) \neq 0$ for every $\boldsymbol{x} \in M$). Let $\boldsymbol{c}$ be a point on $f(M)$, $\boldsymbol{p}_0$ a point such that $\boldsymbol{f}(\boldsymbol{p}_0) = \boldsymbol{c}$, C the connected component of $f^{-1}(\boldsymbol{c})$ containing $\boldsymbol{p}_0$, and suppose $\operatorname{rank}(\boldsymbol{f}'(\boldsymbol{p})) = n-1$ for all $\boldsymbol{p} \in C$ (i.e., $f_1, \ldots, f_{n-1}$ are functionally independent on C). Let $\boldsymbol{\varphi}\colon J \to M$ be the maximal solution of the IVP*

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x}(t_0) = \boldsymbol{p}_0.$$

Then $\boldsymbol{\varphi}(J) = C$, i.e. $x_i = \varphi_i(t)$, $i = 1, \ldots, n$, $t \in J$ is a parameterisation of C, where φ_i is the i^{th} component of $\boldsymbol{\varphi}$.

Proof. From Lemma 1.1.2, and the fact that $\boldsymbol{\varphi}(J)$ is connected, it is clear that $\boldsymbol{\varphi}$ takes values in C. Indeed, if $\boldsymbol{c} = (c_1, \ldots, c_{n-1})$ and if $S_i = \varphi^{-1}(c_i)$, then C is the connected component of $\cap_{i=1}^{n-1} S_i$ containing $\boldsymbol{p}_0$, and we know from Lemma 1.1.2 that $\boldsymbol{\varphi}$ takes values in $\cap_{i=1}^{n-1} S_i$.

We also know, via the Implicit Function Theorem, or more precisely, by the version of the theorem in problems 5 and 6 of HW 7 of ANA2, that C is a one-dimensional manifold. Let $\mathscr{A} = \{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Sigma}$ be an atlas for C. Without loss of generality, we may assume each U_α is connected.

Fix $\alpha \in \Sigma$. Let $I = \psi_\alpha(U_\alpha)$. Then I is an open interval in $\mathbf{R}$. Let $\boldsymbol{\gamma}\colon I \to U_\alpha$ be the inverse of ψ_α, and as usual, write $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$. Then $x_i(\lambda) = \gamma_i(\lambda)$ is a parameterisation of $U_\alpha \subset C$ over the interval I. Since $\psi_\alpha \circ \boldsymbol{\gamma}$ is the identity on I, by the chain rule it follows that $\boldsymbol{\gamma}'(\lambda) \neq \mathbf{0}$ for any $\lambda \in I$.

Next, since $\boldsymbol{\gamma}$ takes values in C, we have $\boldsymbol{f} \circ \boldsymbol{\gamma}$ is constant on I, and the constant value is $\boldsymbol{c}$. By the chain rule we therefore have that

$$\boldsymbol{f}'(\boldsymbol{\gamma}(\lambda))\boldsymbol{\gamma}'(\lambda) = 0, \qquad (\lambda \in I).$$

On the other hand, we also know that $\sum_{k=1}^n v_k \frac{\partial f_i}{\partial x_k} = \boldsymbol{v}(f_i) \equiv 0$ on M for $i = 1, \ldots, n-1$, which implies that $\boldsymbol{f}'(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in M$. Thus both $\boldsymbol{\gamma}'(\lambda)$ and $\boldsymbol{v}(\boldsymbol{\gamma}(\lambda))$ are non-zero vectors in the null space of $\boldsymbol{f}'(\boldsymbol{\gamma}(\lambda))$ for every λ in I. Since the rank of $\boldsymbol{f}'(\boldsymbol{p})$ is $n-1$ for every point of C, the null space of $\boldsymbol{f}'(\boldsymbol{\gamma}(\lambda))$ has dimension one for every $\lambda \in I$. It follows that we have non-zero scalars $u(\lambda)$, one for each $\lambda \in I$, such that

$$(*) \qquad \boldsymbol{\gamma}'(\lambda) = u(\lambda)\boldsymbol{v}(\boldsymbol{\gamma}(\lambda)), \qquad (\lambda \in I).$$

Let $\boldsymbol{p} \in \boldsymbol{\varphi}(J)$, and $s \in J$ a pre-image of $\boldsymbol{p}$. There is an index $\alpha \in \Sigma$ such that $\boldsymbol{p} \in U_\alpha$, and since $\boldsymbol{\varphi}^{-1}(U_\alpha)$ is open and contains s, there is an $\varepsilon > 0$ such that $(s-\varepsilon,\, s+\varepsilon) \subset \boldsymbol{\varphi}^{-1}(U_\alpha)$. In particular, $\psi_\alpha \circ \boldsymbol{\varphi}$ makes sense on $(s-\varepsilon,\, s+\varepsilon)$. Let $\boldsymbol{\gamma}$ and u be as above for this α. For $t \in (s-\varepsilon,\, s+\varepsilon)$ we have, via the chain rule and $(*)$ above,

$$\begin{aligned}
(\psi_\alpha \circ \boldsymbol{\varphi})'(t) &= \psi_\alpha'(\boldsymbol{\varphi}(t))\boldsymbol{v}(\boldsymbol{\varphi}(t)) \\
&= \frac{1}{u(\psi_\alpha(\boldsymbol{\varphi}(t)))}\psi_\alpha'(\boldsymbol{\varphi}(t))\boldsymbol{\gamma}'(\psi_\alpha(\boldsymbol{\varphi}(t))) \\
&= \frac{1}{u(\psi_\alpha(\boldsymbol{\varphi}(t)))}(\psi_\alpha \circ \boldsymbol{\gamma})'(\psi_\alpha(\boldsymbol{\varphi}(t))) \\
&= \frac{1}{u(\psi_\alpha(\boldsymbol{\varphi}(t)))},
\end{aligned}$$

which is non-zero. This means that $\psi_\alpha \circ \boldsymbol{\varphi}$ is a homemorphism on $(s-\varepsilon,\, s+\varepsilon)$. Hence $\boldsymbol{\varphi}$ is a local homeomorphism to C. In particular $\boldsymbol{\varphi}(J)$ is open in C.

In order to show that $\boldsymbol{\varphi}(J) = C$ we claim that it is enough to show the following:

(P) *If $\alpha \in \Sigma$ is such that $U_\alpha \cap \boldsymbol{\varphi}(J) \neq \emptyset$ then $U_\alpha \subset \boldsymbol{\varphi}(J)$.*

Indeed, suppose (P) is true. In that case, if $U_\alpha \not\subseteq \boldsymbol{\varphi}(J)$ then $U_\alpha \cap \boldsymbol{\varphi}(J) = \emptyset$. Let $\Sigma' = \{\alpha \in \Sigma \mid U_\alpha \not\subseteq \boldsymbol{\varphi}(J)\}$ and let $V = \bigcup_{\alpha \in \Sigma'} U_\alpha$. Then $\boldsymbol{\varphi}(J) \cup V = C$, and since $V \cap \boldsymbol{\varphi}(J) = \emptyset$, and since C is connected, either $\boldsymbol{\varphi}(J)$ or V is empty. Now $\boldsymbol{\varphi}(J)$ is clearly non-empty, and hence $V = \emptyset$, whence $C = \boldsymbol{\varphi}(J)$.

Suppose $U_\alpha \cap \boldsymbol{\varphi}(J) \neq \emptyset$. We have to show that $U_\alpha \subset \boldsymbol{\varphi}(J)$. As before, let $I = \psi_\alpha(U_\alpha)$ so that I is an open interval in $\mathbf{R}$, and as before, let $\boldsymbol{\gamma}\colon I \to U_\alpha$ be the inverse of ψ_α. Let $u\colon I \to \mathbf{R}$ be as in $(*)$. For $\lambda^* \in I$ neither $\boldsymbol{\gamma}'(\lambda^*)$ nor $\boldsymbol{v}(\boldsymbol{\gamma}(\lambda^*))$ is zero, and hence there is an index i such that the i^{th} component of both is non-zero. By continuity of $\boldsymbol{\gamma}'$ and $\boldsymbol{v} \circ \boldsymbol{\gamma}$ this property propagates to a neighbourhood I^* of λ^*, whence $u(\lambda) = (\gamma'(\lambda))/(v_i(\lambda))$ on I^*. This proves that $u\colon I \to \mathbf{R}$ is continuous.

Pick $\boldsymbol{p}_1 \in U_\alpha \cap \boldsymbol{\varphi}(J)$. Let $t_1 \in J$ be such that $\boldsymbol{\varphi}(t_1) = \boldsymbol{p}_1$. Let $\lambda_1 = \psi_\alpha(\boldsymbol{p}_1)$. Let

$$t = t_1 + \int_{\lambda_1}^{\lambda} u(y)\mathrm{d}y, \qquad \lambda \in I.$$

Since u is nowhere vanishing and continuous on I, t (in the expression above) is an invertible differentiable function of λ and $\frac{\mathrm{d}t}{\mathrm{d}\lambda} = u(\lambda)$. More formally, let the above map be denoted θ, so that $\theta(\lambda)$ is given by the expression on the right for $\lambda \in I$. Then $\theta(\lambda_1) = t_1$. By the inverse function theorem θ is a diffeomorphism on to its image $J_1 = \theta(I)$, and J_1 is an open interval containing t_1. Let $\xi = \theta^{-1}$ on J_1. Then $\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = u(\lambda)$ and $\frac{\mathrm{d}\xi}{\mathrm{d}t} = (u(\xi(t)))^{-1}$. Define

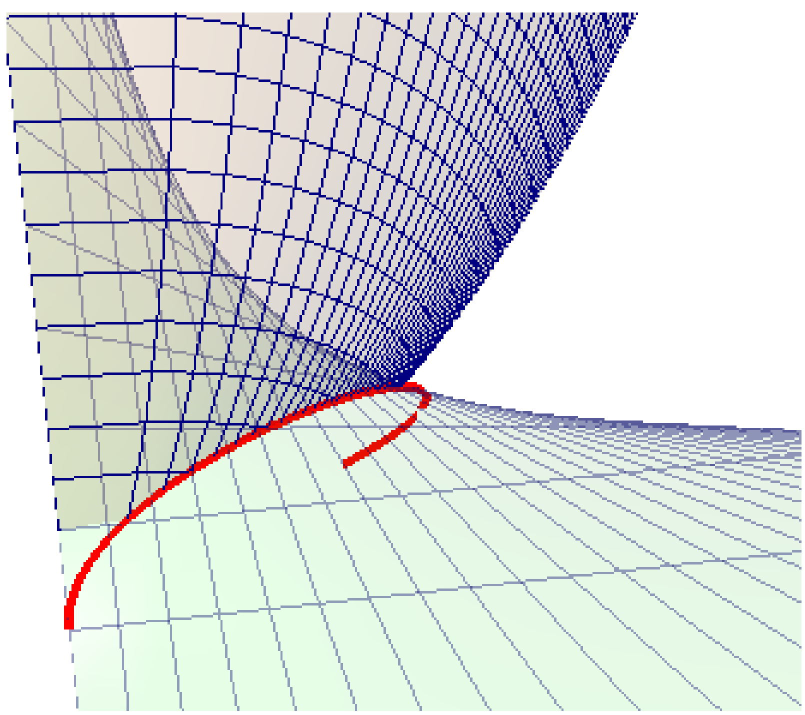
$$\boldsymbol{\varphi}_1 \colon J_1 \to C$$

by the formula $\boldsymbol{\varphi}_1 = \boldsymbol{\gamma} \circ \xi$. Then

$$\begin{aligned}
\dot{\boldsymbol{\varphi}}_1(t) &= \boldsymbol{\gamma}'(\xi(t))\dot{\xi}(t) \\
&= \dot{\xi}(t)u(\xi(t))\boldsymbol{v}(\boldsymbol{\gamma}(\xi(t))) \\
&= \frac{1}{u(\xi(t))}u(\xi(t))\boldsymbol{v}(\boldsymbol{\varphi}_1(t))) \\
&= \boldsymbol{v}(\boldsymbol{\varphi}_1(t)).
\end{aligned}$$

Thus $\boldsymbol{\varphi}_1$ is a solution of $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$. Moreover, $\boldsymbol{\varphi}_1(t_1) = \boldsymbol{p}_1$. Since J is the maximal interval of existence for the IVP $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}(t_1) = \boldsymbol{p}_1$, it follows that $J_1 \subset J$, and $\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}|_{J_1}$, which means $\boldsymbol{\varphi}(J_1) \subset \boldsymbol{\varphi}(J)$. Thus $U_\alpha = \boldsymbol{\gamma}(I) = \boldsymbol{\varphi}_1(J_1) = \boldsymbol{\varphi}(J_1)$ is a subset of $\boldsymbol{\varphi}(J)$. □

The following picture shows the intersection of two level surfaces of first integrals which are functionally independent.



References

[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, translated by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.