LECTURE 14

Date of Lecture: February 17, 2021

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \ldots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \;\|_2$ and we will simply denote it as $\| \;\|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\| \;\|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n, \mathbf{K}^n)$.

☡ Note that $(x_1, \ldots, x_n) \neq [x_1 \ldots x_n]$. Each side is the transpose of the other.

1. First integrals

See also [A1, Chap. 2, § 11] for this material.

1.1. Let $\boldsymbol{v}$ be a smooth vector field on a manifold. A *first integral* for $\boldsymbol{v}$ is a smooth function f on an open set in M such that f is not constant on any open subset of its domain and $\boldsymbol{v}(f) = 0$. Here, we are thinking of $\boldsymbol{v}(p)$ as a derivation at $p \in M$, and $\boldsymbol{v}(f)$ as a function on the domain of f such that its value at p is $\boldsymbol{v}(p)(f)$.

Note that if M is a domain in $\mathbf{R}^n$, and we regard $\boldsymbol{v}$ as a smooth map from M to $\mathbf{R}^n$, say $\boldsymbol{v} = (v_1, \ldots, v_n)$ for some collection of smooth $\mathbf{R}$-valued functions v_i on M, then $\boldsymbol{v}(f) = \sum_i v_i \frac{\partial f}{\partial x_i} = \nabla(f) \bullet \boldsymbol{v}$.[2] Thus, if f is a first integral for $\boldsymbol{v}$, then $\nabla(f)$ and $\boldsymbol{v}$ are orthogonal. Let f be a first integral for f, p be a point in the domain of f and let $c = f(p)$. Let S be the level hypersurface whose equation is

[1] See §§2.1 of Lecture 5 of ANA2.

[2] Here ∇f is the *gradient of* f, i.e., $\nabla(f)$ is the vector field $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ defined on the domain of f.

$f(x_1, \ldots, x_n) = c$ (in other words $S = f^{-1}(f(p))$). Since $\nabla f(p)$ is orthogonal to the level hypersurface S therefore $\boldsymbol{v}(p)$ is tangential to S at p. Moreover, f is constant along any phase curve of $\boldsymbol{v}$.

Example 1.1.1. Let $\boldsymbol{v}$ be the vector field on $\mathbf{R}^3$ given by

$$\boldsymbol{v}(x, y, z) = (x, y, xy(z^2 + 1)).$$

In terms of derivations, $\boldsymbol{v} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + xy(z^2+1)\frac{\partial}{\partial z}$. The corresponding DE is:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ xy(z^2+1) \end{bmatrix}$$

In the open set $R = \{(x, y, z) \mid x \neq 0, y \neq 0\}$ we clearly have

$$\frac{dy}{dx} = \frac{y}{x}.$$

Let us restrict ourselves to one of the four connected components of R, say to $U = \{(x, y, z) \mid x > 0, y > 0\}$. We can solve $\frac{dy}{dx} = \frac{y}{x}$ on U, since it is a separable DE[3] to conclude that

$$\frac{y}{x} = c_1$$

for some constant c_1 (in fact $c_1 = \frac{y_0}{x_0}$ where (x_0, y_0, z_0) is our initial phase). In other words $f \colon U \to \mathbf{R}$ given by

$$f(x, y, z) = \frac{y}{x}$$

is a first integral for $\boldsymbol{v}$. This can also be verified by checking that $x\frac{\partial f}{\partial x}(x, y, z) + y\frac{\partial f}{\partial y}(x, y, z) + xy(z^2+1)\frac{\partial f}{\partial z}(x, y, z) \equiv 0$ on U.

Let us substitute $\frac{y}{x} = c_1$ into the equation $\frac{dz}{dt} = xy(z^2+1)$. We get

$$\frac{dz}{dt} = c_1 x^2 (z^2 + 1).$$

The above, together with $\frac{dx}{dt} = x$, yields (upon taking ratios),

$$\frac{dx}{dz} = \frac{1}{c_1 x (z^2+1)}$$

which is separable and has as solution,

$$c_1 \frac{x^2}{2} = \arctan z + c_2,$$

where c_2 is a constant. Substituting $c_1 = \frac{y}{x}$, we get

$$\frac{1}{2}xy - \arctan z = c_2.$$

Note that $c_2 = \frac{1}{2}x_0 y_0 - \arctan z_0$ where (x_0, y_0, z_0) is our initial phase. Let $g \colon U \to \mathbf{R}$ be the map defined by the formula

$$g(x, y, z) = \frac{1}{2}xy - \arctan z.$$

[3]or just use the fact that $\frac{dy}{dx} = \frac{y}{x}$ implies $\frac{1}{x^2}(x\frac{dy}{dx} - y) = 0$

Then one checks easily that g is a first integral for $\boldsymbol{v}$. We therefore have two first integrals, namely f and g, for our vector field $\boldsymbol{v}$. Moreover,

$$\nabla f = \Big(-\frac{y}{x^2}, \frac{1}{x}, 0\Big), \qquad \nabla g = \Big(\frac{y}{2}, \frac{x}{2}, \frac{-1}{z^2+1}\Big).$$

It is easy to see that these two vectors are never parallel. Or, if you prefer, the Jacobian matrix

$$\frac{\partial(f,g)}{\partial(x,y,z)} = \begin{bmatrix} -\dfrac{y}{x^2} & \dfrac{1}{x} & 0 \\ \\ \dfrac{y}{2} & \dfrac{x}{2} & \dfrac{-1}{z^2+1} \end{bmatrix}$$

has rank 2 at every point of U. Since ∇f and ∇g are never parallel, the level surfaces S_1 and S_2 whose equations are $f(x,y,z) = c_1$ and $g(x,y,z) = c_2$ respectively, are never tangential, i.e., they intersect transversally. According to Problems 5 and 6 of HW7 of ANA2 and Problem 7 of the final exam for ANA2, $S_1 \cap S_2$ is a one-dimensional manifold. A little thought shows that it is a subset of the phase curve giving the solution of our DE.

Let us try to parametrise the curve

$$C = S_1 \cap S_2.$$

A simple elimination gives

$$x = x, \quad y = c_1 x, \quad z = \tan\Big(c_1 \frac{x^2}{2} - c_2\Big).$$

We know that the velocity vectors for this parameterised curve will be multiples of $\boldsymbol{v}$ at each point of C, because for any $p \in C$, the velocity vector of the above parameterised curve, as well as the vector $\boldsymbol{v}(p)$ will be tangential to C at p. However, we wish these two vectors to coincide, and this may require re-parameterisation. Recall,

$$\frac{dx}{dt} = x.$$

This gives us $t = \ln x - \ln x_0$ as a way of marking time. The corresponding parameterisation is:

$$(x,y,z) = \Bigg(x_0 e^t, c_1 x_0 e^t, \tan\Big(\frac{1}{2} c_1 x_0^2 e^{2t} - c_2\Big)\Bigg).$$

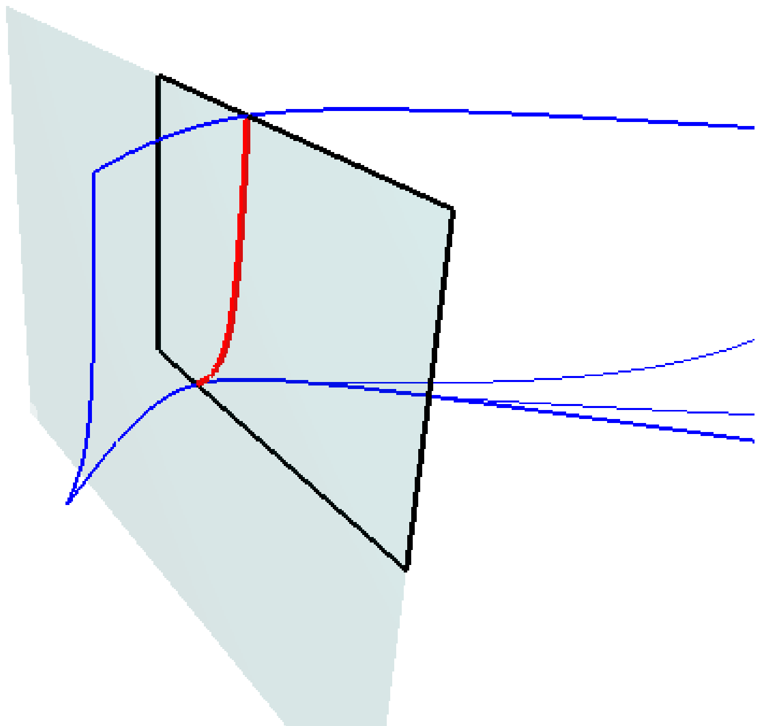
Let $\boldsymbol{p_0} = (x_0, y_0, z_0)$ be the initial phase, and $\boldsymbol{\varphi_{p_0}}$ the phase curve through $\boldsymbol{p_0}$ such that $\boldsymbol{\varphi_{p_0}}(0) = \boldsymbol{p_0}$. Substituting $c_1 = y_0/x_0$ and $c_2 = \frac{1}{2}x_0 y_0 - \arctan z_0$ into the above matrix identity, we get

$$\boldsymbol{\varphi_{p_0}}(t) = \Bigg(x_0 e^t, y_0 e^t, \tan\Big(\frac{1}{2} x_0 y_0 e^{2t} - \frac{1}{2} x_0 y_0 + \arctan z_0\Big)\Bigg).$$

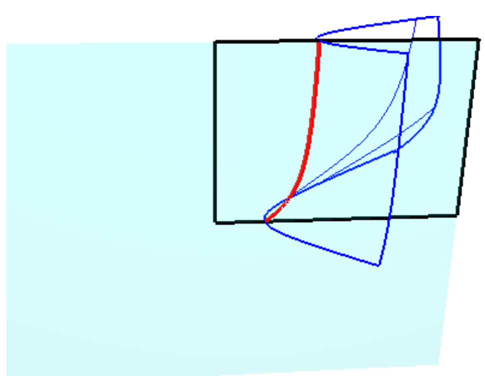
Seen this way, it is obvious that the solution $\boldsymbol{\varphi_{p_0}}$ is smooth with respect to the initial point $\boldsymbol{p_0} = (x_0, y_0, z_0)$ — as predicted by the theory of ODE's. The maximal interval of existence for the solution when the initial phase is $\boldsymbol{p}_0$ is

$$J_{\max}(\boldsymbol{p}_0) = \{t \in \mathbf{R} \mid -\pi/2 < \frac{1}{2} x_0 y_0 e^{2t} - \frac{1}{2} x_0 y_0 + \arctan z_0 < \pi/2\}.$$

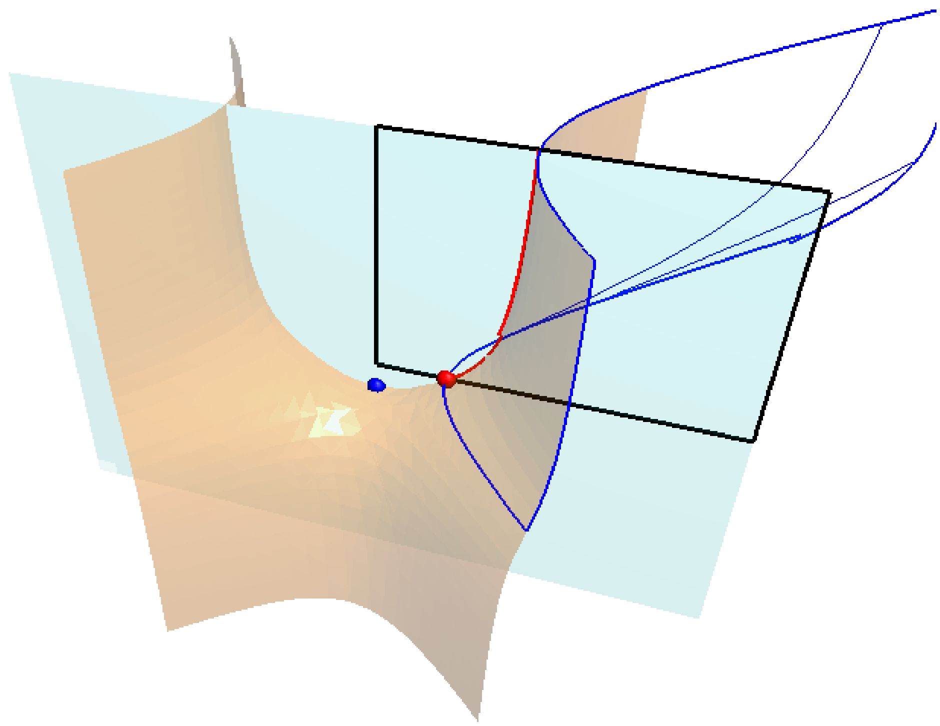
Question: Does the same formula work for other connected components of $R = \{(x,y,z) \mid x \neq 0, y \neq 0\}$? What about the origin? Here is a picture of the level surfaces $f(x,y,z) = 1$ and $g(x,y,z) = \frac{1}{2}$ and their intersection curve. Note that $f = 1$ is the plane $x = y$. The intersection is the phase curve through $(1,1,0)$.



Here is another angle



And below (the next page actually) is an angle with a fuller picture of the surface $\frac{1}{2}xy - \arctan z = \frac{1}{2}$. The big red dot is the initial phase $(1,1,0)$. I have plotted the forward evolution until the trajectory hits $z = 5$. The saddle point (blue dot) that you can see on the level surface $g = \frac{1}{2}$ is the point $(0,0,-\tan(\frac{1}{2}))$. From the picture it does seem as if a backward evolution past this point is possible. Is that really so? Think about it.



References

[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, translated by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.