LECTURE 13

Date of Lecture: February 15, 2021

As always, $\mathbf{K} \in \{\mathbf{R},\, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\|\;\|_2$ and we will simply denote it as $\|\;\|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\|\;\|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n,\, \mathbf{K}^n)$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. **Manifolds**

1.1. Recall that for us an n-dimensional manifold is a smooth differentiable manifold. This means M is a Hausdorff, second countable topological space together with data (called an *atlas*) $\mathscr{A}$, where is $\mathscr{A}$ is a collection of pairs of the form $(U,\, \varphi)$ with U an open subset of M, $\varphi\colon U \to \mathbf{R}^n$ an open map which is a homeomorphism on to its image (equivalently, $\varphi(U)$ is open in $\mathbf{R}^n$ and $\varphi\colon U \to \varphi(U)$ is a homeomorphism) such that

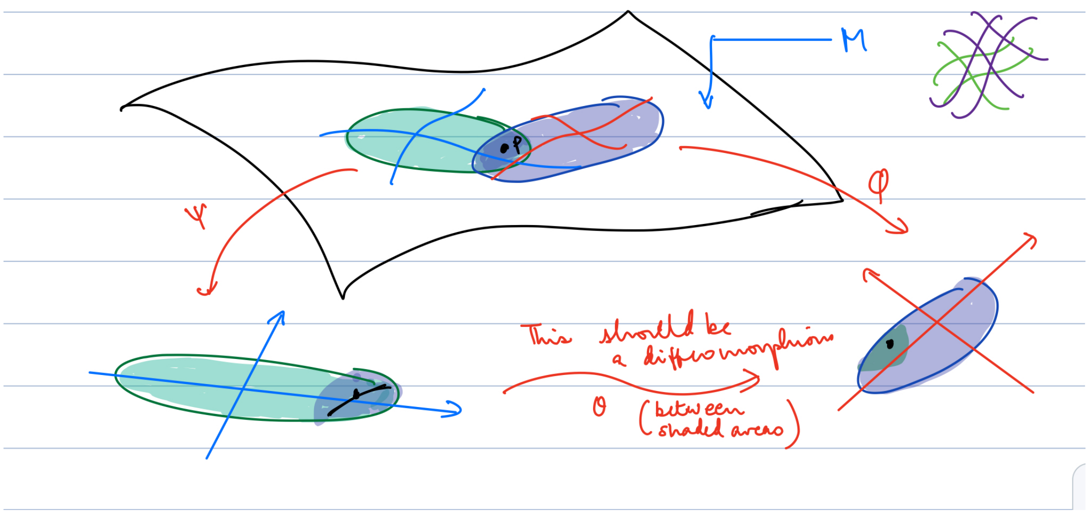
- $M = \bigcup_{(U,\varphi)\in\mathscr{A}} U$.

[1] See §§2.1 of Lecture 5 of ANA2.

- If (U, φ) and (V, ψ) are members of $\mathscr{A}$ then the map θ defined by the commutativity of the diagram below is a diffeomorphism:

$$\begin{tikzcd} & U \cap V \arrow[dl, "\text{via } \psi"', "\sim"] \arrow[dr, "\text{via } \phi", "\sim"'] & \\ \psi(U \cap V) \arrow[rr, "\sim", "\theta"'] & & \phi(U \cap V) \end{tikzcd}$$

The situation is often represented by pictures of the following sort:



A collection $\mathscr{A}$ satisfying the requirements above is called an *atlas*, and a member (U, φ) of $\mathscr{A}$ is called a *co-ordinate chart*. The map θ is called a transition map or transition function and is often denoted as θ_{UV} or $\theta_{\varphi\psi}$. Two atlases $\mathscr{A}$ and $\mathscr{A}'$ on M are said to endow M with the same differentiable structure if there exists a third atlas $\mathscr{A}''$ such that both $\mathscr{A}$ and $\mathscr{A}'$ are contained in $\mathscr{A}''$. Given an atlas $\mathscr{A}$ there is a unique maximal atlas $\mathscr{A}_{\max}$ containing $\mathscr{A}$. Technically a differentiable manifold is a topological space M together with a maximal atlas. But as the remarks above show, the differentiable manifold structure on M is fully determined by any atlas, not necessarily maximal. The difference is one of semantics.

1.2. **The tangent bundle.** In the notes on vector fields we defined the notion of the tangent bundle

$$\begin{tikzcd} T(M) \arrow[d, "\varpi"] \\ M \end{tikzcd}$$

to a manifold M. For any $p \in M$, $\varpi^{-1}(p) = T_p(M)$ is the tangent space of M at p and hence a vector space over $\mathbf{R}$. In greater detail, $T_p(M) = \mathscr{D}\mathrm{er}_p$, the vector space of derivations on the $\mathbf{R}$-algebra $\mathscr{C}_p^\infty$ of germs of $\mathscr{C}^\infty$ functions at p. Thus $D \in T_p(M)$ if it is an $\mathbf{R}$-linear map $D\colon \mathscr{C}_p^\infty \to \mathbf{R}$ satisfying $D(fg) = f(p)Dg + g(p)Df$.

Fix a manifold M. We wish to make sense of differential equations of the form $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$ (that we have on open subsets of $\mathbf{R}^{n+1}$) in the more general setting of manifolds. What follows are the ingredients needed to make the transition from open subsets of $\mathbf{R}^{n+1}$ to open subsets of $\mathbf{R} \times M$.

Given a $\mathscr{C}^1$ path $\gamma\colon I \to M$, with I an open interval, and t a point in I, set $\dot{\gamma}(t)$ equal to the derivation on $\mathscr{C}^\infty_{\gamma(t)}$ given by

$$(\dot{\gamma}(t))(f) = \frac{\mathrm{d}}{\mathrm{d}s}(f(\gamma(s)))\Big|_{s=t} \qquad (f \in \mathscr{C}^\infty_{\gamma(t)}).$$

We point out that the expression on the right makes sense for germs of $\mathscr{C}^\infty$ functions at $\gamma(t)$, for if we replace the germ f by a representative in a neighbourhood of $\gamma(t)$, the derivative on the right can be computed and evaluated at $s = t$, and the result is independent of the representative chosen. Thus $\dot{\gamma}(t) \in T_{\gamma(t)}(M)$ for $t \in I$. We thus have a map $\dot{\gamma}\colon I \to T(M)$. Since γ is $\mathscr{C}^1$, it is not hard to see, using coordinates, that $\dot{\gamma}$ is continuous. Moreover the following diagram commutes

$$\begin{tikzcd} & T(M) \arrow[d, "\varpi"] \\ I \arrow[ur, "\dot{\gamma}"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

The generalisation of our map $\boldsymbol{v}$ (in the classical form of the differentials equations we have been considering) is a *vector field* on M which was defined in 3.1.4 of the supplementary note on vector fields. Recall that a vector field on M is a section v of $\varpi\colon T(M) \to M$. in other words $v\colon M \to T(M)$ is a map such that $\varpi \circ v = \mathbf{1}_M$.

$$\begin{tikzcd} T(M) \arrow[d, "\varpi"] \\ M \arrow[u, bend left, "v"] \end{tikzcd}$$

The vector field v is $\mathscr{C}^p$ if the map is v is $\mathscr{C}^p$ in local Euclidean coordinates.

If v is a vector field on M then the generalisation of an autonomous DE is an equation of the form

$$\dot{x} = v(x).$$

We assume v is continuous (and much more often v will be $\mathscr{C}^1$). A solution is a $\mathscr{C}^1$ path $\varphi\colon I \to M$ on an open interval I such that $\dot{\varphi}(t) = v(\varphi(t))$ for $t \in I$, where the last equality is an equality of points in $T(M)$. In other words we require φ to fit into the commutative diagram below.

$$\begin{tikzcd} & T(M) \arrow[dr, "\varpi"] & \\ I \arrow[ur, "\dot{\varphi}"] \arrow[r, "\varphi"'] & M \arrow[u, "v"] \arrow[r, equal] & M \end{tikzcd}$$

If $p \in M$ and $\tau \in \mathbf{R}$ we can clearly talk about the IVP

$$\dot{x} = v(x), \qquad x(\tau) = p.$$

A solution to the above IVP is a solution $\varphi\colon I \to M$ to the underlying DE such that $\tau \in I$ and $\varphi(\tau) = p$.

What about the non-autonomous case? Let Ω be an open subset of $\mathbf{R} \times M$. Let $\pi\colon \Omega \to M$ be the second projection. We regard Ω as an extended phase space. Let $v\colon \Omega \to T(M)$ be a continuous map such that $\varpi \circ v = \pi$. In other words we have a commutative diagram;

$$\begin{tikzcd} & T(M) \arrow[d, "\varpi"] \\ \Omega \arrow[ur, "v"] \arrow[r, "\pi"'] & M \end{tikzcd}$$

The corresponding DE is

$$\dot{x} = v(t, x).$$

A solution is a $\mathscr{C}^1$ map $\varphi\colon I \to M$ on an open interval I such that $(t, \varphi(t)) \in \Omega$ for all $t \in I$ and $\dot{\varphi}(t) = v(t, \varphi(t))$ for $t \in I$, where (as in the autonomous case) the last equality is an equality of points in $T(M)$. This means that φ is required to fit into the commutative diagram (with $\widetilde{\varphi}\colon I \to \Omega$ the map $t \mapsto (t,\, \varphi(t))$):

$$\begin{tikzcd} & T(M) \arrow[dr, "\varpi"] & \\ I \arrow[ur, "\dot{\varphi}"] \arrow[r, "\widetilde{\varphi}"] \arrow[rr, bend right, "\varphi"'] & \Omega \arrow[u, "v"] \arrow[r, "\pi"] & M \end{tikzcd}$$

It is clear that if $(\tau, p) \in \Omega$ is a fixed point then all this can be upgraded to an IVP of the form

$$\dot{x} = v(t, x), \qquad x(\tau) = p.$$

Solutions of such IVP's can only mean one thing. They are solutions φ of the underlying DE such that the domain of φ contains τ and $\varphi(\tau) = p$.

1.2.1. It is evident that if v is $\mathscr{C}^1$ then solutions to the IVP above exist, that there is a maximal interval of existence on which a solution exists, that this solution on the maximal interval is unique, every interval of existence for the IVP is a subset of this maximal interval, and every solution of the IVP is the restriction of the solution on the maximal interval of existence. All of this is immediate from what we have proved in the classical case when Ω is a domain in $\mathbf{R}^{n+1}$ and the fact that on Euclidean domains $\mathscr{C}^1$ maps are locally Lipschitz in the second argument.

References

[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, translated by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.