LECTURE 12

Date of Lecture: February 10, 2021

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \, \|_2$ and we will simply denote it as $\| \, \|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\| \, \|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n, \mathbf{K}^n)$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. Applications of real canonical forms to Linear ODEs

See [G, pp.37–40], [CL, Chapter 3] and [A1, Chapter3, § 25] for other material on this topic.

1.1. Exponentials.

The following lemma is useful for "computing" exponentials via Jordan decompositions.

Lemma 1.1.1. *Let $A \in M_n(\mathbf{R})$, $\Gamma \in GL_n(\mathbf{R})$.*

(a) $e^{\Gamma A \Gamma^{-1}} = \Gamma e^A \Gamma^{-1}$.

[1] See §§2.1 of Lecture 5 of ANA2.

(b) If

$$A = \begin{bmatrix} A_1 & 0 & & & 0 \\ & A_2 & 0 & & 0 \\ & & A_3 & \ddots & \\ & & & \ddots & 0 \\ & & & & A_t \end{bmatrix}$$

then

$$e^A = \begin{bmatrix} e^{A_1} & 0 & & & 0 \\ & e^{A_2} & 0 & & 0 \\ & & e^{A_3} & \ddots & \\ & & & \ddots & 0 \\ & & & & e^{A_t} \end{bmatrix}$$

Proof. For part (a), for each non-negative integer N we have

$$\sum_{n=0}^{N} \frac{(\Gamma A \Gamma^{-1})^n}{n!} = \Gamma \left(\sum_{n=0}^{N} \frac{A^n}{n!} \right) \Gamma^{-1}.$$

From item **8** in §1 of [ANA2, Lecture 7] , the product of matrices respects limits. Let $N \to \infty$ in the above to get (a).

For (b), let B_i be the block diagonal $n \times n$ matrix in which the A_k in A are replaced by 0, for $k \neq i$, and the i^{th} block is A_i. Then $B_1, \dots, B_t$ commute and their sum is A. Therefore, we may assume, without loss of generality that $A_k = 0$ for $k \geq 2$. Let the size of each block in the decompositions we are considering be $r_1, \dots, r_t$. Let

$$Z_N = \sum_{n=0}^{N} \frac{A_1^n}{n!} - e^{A_1}.$$

Set

$$E = \begin{bmatrix} e^{A_1} & 0 & & & 0 \\ & I_{r_2} & 0 & & 0 \\ & & I_{r_3} & \ddots & \\ & & & \ddots & 0 \\ & & & & I_{r_t} \end{bmatrix}$$

Then

$$\sum_{n=0}^{N} \frac{A^n}{n!} - E = \begin{bmatrix} Z_N & 0 & & & 0 \\ & 0 & 0 & & 0 \\ & & 0 & \ddots & \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix}$$

Moreover, it is easy to see that the $\| \cdot \|_\circ$ of the matrix on the right is actually $\|Z_N\|_\circ$. Now, $Z_N \to 0$ as $N \to \infty$, and hence $\sum_{n=0}^{\infty} \frac{A^n}{n!} = E$. Thus $e^A = E$, which is what we had to show. □

1.2. **Structure of solutions of homogeneous linear equations.** Let A be a constant $n \times n$ matrix, i.e., $A \in M_n(\mathbf{R})$. Consider the associated DE

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}. \tag{1.2.1}$$

Let the Jordan form of A be

$$\mathbf{J} = \begin{bmatrix} \widetilde{J}_1 & 0 & 0 & \dots & 0 \\ & \widetilde{J}_2 & 0 & \dots & 0 \\ & & \ddots & & \\ & & & \widetilde{J}_{t-1} & 0 \\ & & & & \widetilde{J}_t \end{bmatrix}.$$

There exists $\Gamma \in GL_n(\mathbf{R})$ such that

$$A = \Gamma \mathbf{J} \Gamma^{-1}.$$

The block $\widetilde{J}_k$ are either of the form [Lecture 11, (1.1.2)] (for real eigenvalues λ) or of the form [Lecture 11, (1.1.5)] with M of the form [Lecture 11, (1.1.6)], for eigenvalues $\lambda = a + ib$ with $b \neq 0$. Since solutions of (1.2.1) are of the form $e^{tA}\boldsymbol{x}_0$, $\mathbf{x_0} \in \mathbf{R^n}$, we can apply Lemma 1.1.1 to work out the solutions.

If $\widetilde{J}_k$ is of the form [Lecture 11, (1.1.2)] with the diagonal entries being $\lambda_k \in \mathbf{R}$, then writing $\widetilde{J}_k = \lambda_k I_{r_k} + B$, and using results from Quiz 2, (note λI_{r_k} and B commute) we see that

$$e^{t\widetilde{J}_k} = e^{t\lambda_k} \begin{bmatrix} 1 & t & t^2/2! & \dots & t^{r_k-1}/(r_k-1)! \\ & 1 & t & \dots & t^{r_k-2}/(r_k-2)! \\ & & 1 & \dots & t^{r_k-3}/(r_k-3)! \\ & & & \ddots & t^2/2! \\ & & & & t \\ & & & & 1 \end{bmatrix}.$$

If $\widetilde{J}_k$ is of the form [Lecture 11, (1.1.5)] with M of the form [Lecture 11, (1.1.6)] with $a = a_k$ and $b = b_k$, then we know from Problem **4)** of HW 5 that

$$e^{t\widetilde{J}_k} = e^{a_k t} \begin{bmatrix} B & tB & t^2/2!B & \dots & t^{r_k-1}/(r_k-1)!B \\ & B & tB & \dots & t^{r_k-2}/(r_k-2)!B \\ & & B & \dots & t^{r_k-3}/(r_k-3)!B \\ & & & \ddots & t^2/2!B \\ & & & & tB \\ & & & & B \end{bmatrix}.$$

where

$$B = \begin{bmatrix} \cos(b_k t) & \sin(b_k) \\ -\sin(b_k t) & \cos(b_k t) \end{bmatrix}$$

Using Lemma 1.1.1 we see that solutions are of the form $\boldsymbol{\varphi} \colon \mathbf{R} \to \mathbf{R}^n$ where $\varphi_k(t)$ is a linear combination of $\{t^i e^{\lambda_k t} \mid i = 0, \dots, r_k - 1, \lambda_k \in \mathbf{R}\}$, $\{t^i e^{a_k t} \cos b_k t \mid i = 0, \dots, r_k - 1, \lambda_k = a_k + ib_k, b_k \neq 0\}$ and $\{t^i e^{a_k t} \sin b_k t \mid i = 0, \dots, r_k - 1, \lambda_k = a_k + ib_k, b_k \neq 0\}$, as k ranges from 1 to t. We are not claiming that every possible linear combination is possible for each entry each independent of other entires. That would give n^2 degrees of freedom for the number of solutions. Moreover, two distinct real Jordan blocks $\widetilde{J}_k$ and $\widetilde{J}_l$ may well have the same (real or complex) associated eigenvalue(s).

1.3. **Scalar n^{th} order Linear Differential Equations.** Consider the scalar n^{th} order linear DE with constant coefficiants.

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_i y' + a_n y = 0. \tag{1.3.1}$$

As is well-known this is equivalent to the first order linear autonomous DE of the form

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}. \tag{1.3.2}$$

$$A = \begin{bmatrix} 0 & 1 & & \ddots & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ -a_0 & -a_1 & & \dots & -a_{n-1} \end{bmatrix}$$

If $\boldsymbol{\psi}\colon \mathbf{R} \to \mathbf{R}^n$ is a solution of (1.3.2), then $\varphi = \psi_1$ is a solution of (1.3.1). Conversely, if $\varphi\colon \mathbf{R} \to \mathbf{R}$ is a solution of (1.3.1), then $\boldsymbol{\psi}\colon \mathbf{R} \to \mathbf{R}^n$ with $\psi_i = \varphi^{(i-1)}$, $i = 1, \dots, n$, is a solution of (1.3.2) where $\varphi^{(0)} := \varphi$. Moroever, the characteristic polynomial of A is $\pm$ the characteristic polynomial of (1.3.1) according to Problem **1)** of HW 5. Let

$$\sigma(A) = \{\lambda \in \mathbf{C} \mid \lambda \text{ is an eigenvalue of } A\},$$

and

$$S(A) = \sigma(A)/R,$$

where R is the equivalence relation $\lambda R \tau$ if either $\sigma = \tau$ or $\bar{\sigma} = \tau$. Note that we have a partition

$$\sigma(A) = \sigma_1(A) \sqcup \sigma_2(A)$$

where $\sigma_1(A) = \sigma(A) \cap \mathbf{R}$, and $\sigma_2(A) = \sigma(A) \setminus \sigma_A(A)$. Under R, each element of $\sigma_1(A)$ is an equivalence class by itself, whereas, the equivalence classes of elements of $\sigma_2(A)$ consist of two elements, $\{\lambda, \bar{\lambda}\}$. In the same way $S(A)$ partitions into

$$S(A) = S_1(A) \sqcup S_2(A).$$

According to Problem **2)** of HW5 there is a one-to-one correspondence between the number of Jordan blocks of A and $\sigma(A)$, because of the special form of A associated with (1.3.2) and (1.3.1). Our discussion on real Jordan forms then shows that there is a one-to-one correspondence between $S(A)$ and the number of real Jordan blocks.

Let $[\lambda]$ denote the R-equivalence class of $\lambda \in \sigma(A)$. If $[\lambda] \in S_1(A)$, then the real Jordan form associated with $[\lambda]$ is as in [Lecture 11, (1.1.2)] the size being equal to the multiplicity of the real root λ of the characteristic polynomial of A. If $\lambda = a+ib \in \sigma_2(A)$, with $b > 0$ for definiteness, then the real Jordan form associated with $[\lambda] \in S_2(A)$ is as in [Lecture 11, (1.1.5)] with M being the matrix in [Lecture 11, (1.1.6)].

Let the Jordan block associated with $s \in S(A)$ be denoted $\widetilde{J}_s$. With each $s \in S(A)$ there is a well defined multiplicity r_s associated with s, namely the multiplicity of any root λ_s of the characteristic equation of A in the equivalence class s. This is half the size of the Jordan block associated with $\widetilde{J}_s$ if $s_k \in S_2(A)$, and equal to the size of the Jordan block $\widetilde{J}_s$ if $s \in S_1(A)$. If $s \in S_2(A)$, let a_s and

b_s be real numbers, with $b_s > 0$ such that $s = [a_s + ib_s]$. Since we are insisting b_ki is positive, it is well-defined.

From our earlier discussion, it follows that the solutions of (1.3.1) are in the linear span of the set

$$Q = \bigcup_{s \in S_1(A)} \bigcup_{j=0}^{r_s - 1} \{t^j e^{t\lambda_s}\} \cup \bigcup_{s \in S_2(A)} \bigcup_{j=0}^{r_s - 1} \{t^j e^{a_s t} \cos b_s t,\ t^j e^{a_s t} \sin b_s t\}.$$

The space of solutions of (1.3.1) is an n-dimensional $\mathbf{R}$-vector space, and the cardinality of $Q \leq n$. It follows that the cardinality of Q is n and Q is a basis for the space of solutions of (1.3.1). In particular, the general solution of (1.3.1) is

$$y(t) = \sum_{s \in S_1(A)} \sum_{j=0}^{r_s - 1} c_{sj} t^j e^{\lambda_s t} + \sum_{s \in S_2(A)} \sum_{j=0}^{r_s - 1} \Big(d_{js} t^j e^{a_s t} \cos b_s t + e_{sj} t^j e^{a_s t} \sin b_s t \Big),$$

with c_{js}, d_{js}, and e_{js} being arbitrary real constants, uniquely determined by the solution $y(t)$.

1.3.3. The above proves the statements made in the section on (scalar) homogeneous linear DE's with constant coefficients in DEQN Cookbook-II. In particular, the union of the sets labelled R, C, and S in *loc.cit.* form a basis for the solution space of such equations, a fact which is not *a priori* obvious.

References

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