LECTURE 11

Date of Lecture: February 8, 2021

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \ldots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \,\|_2$ and we will simply denote it as $\| \,\|$. The space of $\mathbf{K}$-linear transformations from $\mathbf{K}^n$ to $\mathbf{K}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{K}^n, \mathbf{K}^m)$ and will be identified in the standard way with the space of $m \times n$ matrices $M_{m,n}(\mathbf{K})$ and the operator norm[1] on both spaces will be denoted $\| \,\|_\circ$. If $m = n$, we write $M_n(\mathbf{R})$ for $M_{m,n}(\mathbf{R})$, and $L(\mathbf{K}^n)$ for $\mathrm{Hom}_{\mathbf{K}}(\mathbf{K}^n, \mathbf{K}^n)$.

☡ Note that $(x_1, \ldots, x_n) \neq [x_1 \ldots x_n]$. Each side is the transpose of the other.

1. Real canonical forms

I assume you know what Jordan forms are over algebraically closed fields. Since $\mathbf{R}$ is not algebraically closed, a little effort is needed to get real canonical forms. A good reference for this topic is [AW].

1.1. Let $T \in L(\mathbf{R}^n)$ and let $A \in M_n(\mathbf{R})$ be its matrix with respect to the standard basis on $\mathbf{R}^n$, where $M_n(\mathbf{R})$ is the ring of $n \times n$ real matrices. Regarding A as a complex matrix, and letting $T_{\mathbf{C}} \in L(\mathbf{C}^n)$ be the corresponding linear operator on $\mathbf{C}^n$,[2] we know that $\mathbf{C}^n$ has an *ordered* basis $\mathscr{B}$ such that the matrix of T with

[1] See §§2.1 of Lecture 5 of ANA2.

[2] Let $\boldsymbol{u} \in \mathbf{C}^n$ and suppose $\boldsymbol{u} = \boldsymbol{c} + i\boldsymbol{d}$, with $\boldsymbol{c}, \boldsymbol{d} \in \mathbf{R}^n$ be its decomposition into real and imaginary parts. Then $T_{\mathbf{C}}\boldsymbol{u} = T\boldsymbol{c} + iT\boldsymbol{u}$.

respect to $\mathscr{B}$ is a Jordan canonical form, i.e., the matrix of $T_{\mathbf{C}}$ with respect to $\mathscr{B}$ looks like a block diagonal matrix

$$\begin{bmatrix} J_1 & 0 & & & 0 \\ & J_2 & 0 & & 0 \\ & & J_3 & \ddots & \\ & & & \ddots & 0 \\ & & & & J_t \end{bmatrix} \tag{1.1.1}$$

where each block (the so-called Jordan block) has the form.

$$J = \begin{bmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix} \tag{1.1.2}$$

Note that λ has to be an eigenvalue of T (or $T_{\mathbf{C}}$), and every eigenvalue is accounted for in some Jordan block – in perhaps more than one Jordan block. The number of Jordan blocks corresponding to a fixed eigenvalue λ is known to be the geometric multiplicity of λ.

The Jordan decomposition (1.1.1) decomposes $\mathbf{C}^n$ into a direct sum

$$\mathbf{C}^n = V_1 \oplus \cdots \oplus V_t \tag{1.1.3}$$

with V_k corresponding to the Jordan block J_k, $k = 1, \ldots, t$ and J_k can be regarded as a linear operator on V_k. The sets $\mathscr{B} \cap V_k$, $k = 1, \ldots, t$ partition $\mathscr{B}$ into disjoint sets, and for each k, $\mathscr{B} \cap V_k$ is a basis of V_k. In other words, if $\mathscr{B} \cap V_k = \{\boldsymbol{u}_1^k, \boldsymbol{u}_2^k, \ldots, \boldsymbol{u}_{r_k}^k\}$, where we list the elements according to the order in $\mathscr{B}$, then $T_{\mathbf{C}}\boldsymbol{u}_j^k = \sum_{i=1}^{r_k} a_{ij}^{(k)} \boldsymbol{u}_i^k$, where $J_k = (a_{ij}^{(k)})$. Moreover, $\mathscr{B}$ can be so chosen that the conjugates $\bar{\boldsymbol{u}}_i^k$ of $\boldsymbol{u}_i^k$, $j = 1, \ldots, r_k$ are also in $\mathscr{B}$ and form an ordered basis for some V_l in the decomposition (1.1.3) above. If the eigenvalue corresponding to V_k is real, then $V_l = V_k$, otherwise $V_l \neq V_k$, where l and k are related as above.

Let V be one of the V_k, and without loss of generality, assume $V = V_1$. Let J be the corresponding Jordan block, i.e., J is of the form (1.1.2). Assume that the eigenvalue λ corresponding to J is not real, i.e.,

$$\lambda \notin \mathbf{R}.$$

Let $\mathscr{B} \cap V = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r\}$, the ordering of the subscripts respecting the order in $\mathscr{B}$. From the form of J we see that

$$\begin{aligned} T_{\mathbf{C}}(\boldsymbol{u}_1) &= \lambda \boldsymbol{u}_1 \\ T_{\mathbf{C}}(\boldsymbol{u}_i) &= \boldsymbol{u}_{i-1} + \lambda \boldsymbol{u}_i, \quad i = 2, \ldots, r. \end{aligned} \tag{$*$}$$

From what we said above about $\mathscr{B}$, if $\boldsymbol{w}_i := \bar{\boldsymbol{u}}_i$, $i = 1, \ldots, r$ then $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r\}$ is an ordered basis for some V_l occurring in the decomposition (1.1.3). Since $\lambda \notin \mathbf{R}$, we have $V_l \neq V$ and so we may as well assume $l = 2$, and write $W = V_2$. We have

$$\begin{aligned} T_{\mathbf{C}}(\boldsymbol{w}_1) &= \bar{\lambda} \boldsymbol{w}_1 \\ T_{\mathbf{C}}(\boldsymbol{w}_i) &= \boldsymbol{w}_{i-1} + \bar{\lambda} \boldsymbol{w}_i, \quad i = 2, \ldots, r. \end{aligned} \tag{$\dagger$}$$

Note that

$$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_r\}$$

is a basis for $V \oplus W \subset \mathbf{C}^n$.

For $j \in \{1, \ldots, r\}$, let

$$\boldsymbol{c}_j = \frac{1}{2}(\boldsymbol{u}_j + \boldsymbol{w}_j) \quad \text{and} \quad \boldsymbol{d}_j = \frac{1}{2i}(\boldsymbol{u}_j - \boldsymbol{w}_j).$$

Then,

$$\boldsymbol{u}_j = \boldsymbol{c}_j + i\boldsymbol{d}_j \quad \text{and} \quad \boldsymbol{w}_j = \boldsymbol{c}_j - i\boldsymbol{d}_j \qquad (j = 1, \ldots, r).$$

It follows that the span of $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_r\}$ is the same as the span of $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r, \boldsymbol{d}_1, \ldots, \boldsymbol{d}_r\}$. In particular, $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r, \boldsymbol{d}_1, \ldots, \boldsymbol{d}_r\}$ is a basis for $V \oplus W$. We point out that $\boldsymbol{c}_j, \boldsymbol{d}_j \in \mathbf{R}^n$.

If $\lambda = a + ib$, an easy computation shows that

$$T\boldsymbol{c}_j = \begin{cases} a\boldsymbol{c}_j - b\boldsymbol{d}_j & \text{if } j = 1 \\ \boldsymbol{c}_{j-1} + a\boldsymbol{c}_j - b\boldsymbol{d}_j & \text{if } 2 \le j \le r \end{cases}$$

and

$$T\boldsymbol{d}_j = \begin{cases} b\boldsymbol{c}_j + a\boldsymbol{d}_j & \text{if } j = 1 \\ \boldsymbol{d}_{j-1} + b\boldsymbol{c}_j + a\boldsymbol{d}_j & \text{if } 2 \le j \le r. \end{cases}$$

If we modify our basis so that the first $2r$ members are

$$\boldsymbol{c}_1, \boldsymbol{d}_1, \boldsymbol{c}_2, \boldsymbol{d}_2, \ldots, \boldsymbol{c}_r, \boldsymbol{d}_r \tag{1.1.4}$$

and do this for every pair of conjugate eigenvalues which are not real, then the the matrix of T with respect to this basis is such that the the blocks corresponding to segments like (1.1.4) have the form:

$$\widetilde{J} = \widetilde{J}(\lambda, \bar{\lambda}) = \begin{bmatrix} M & I_2 & & & 0 \\ & M & I_2 & & \\ & & M & \ddots & \\ & & & \ddots & I_2 \\ & & & & M \end{bmatrix} \tag{1.1.5}$$

where

$$M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}. \tag{1.1.6}$$

Blocks of the form (1.1.2), when $\lambda \in \mathbf{R}$, or of the form (1.1.5), with M as in (1.1.6) when $\lambda \notin \mathbf{R}$, are called *real Jordan blocks* or *real canonical blocks*. When we rewrite the matrix of T in terms of this basis, the resulting matrix is called the *real Jordan form* or the *the real canonical form* of A. It is a block diagonnal matrix where the blocks are real canonical forms. See [AW, pp. 254–256] for details about real Jordan canonical forms.

References

[AW] W.A. Adkins and A.H. Weintraub, *Agebra, An Approach via Module Theory*, Graduate Texts in Mathematics **136**, Springer-Verlag, New York, 1992.

[A1] V. I. Arnold, *Ordinary Differential Equations*, translated by Richard A. Silverman, MIT Press (also Prentice-Hall, India), Cambridge,MA, U.S.A., 1973.

[A2] V. I. Arnold, *Ordinary Differential Equations*, translated by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2006.