LECTURE 1

Date of Lecture: January 4, 2021

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5 of ANA2.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \, \|_2$ and we will simply denote it as $\| \, \|$. The space of $\mathbf{R}$-linear transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$ will be denoted $\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^m)$ and will be identified in the standard way with the space of $m \times n$ real matrices $M_{m,n}(\mathbf{R})$ and the operator norm[1] on both spaces will be denoted $\| \, \|_\circ$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. **Ordinary Differential Equations**

We give a formulation of the problems we will study in this course.

1.1. **What is an Ordinary Differential Equation?**

Ordinary Differential Equations (ODEs) in their simplest form are equations of the form

$$F(t, u(t), u'(t), \dots, u^{(n)}(t)) = 0 \tag{1.1.1}$$

where $F \colon I \times U \to \mathbf{R}$ us a function, with I an interval in $\mathbf{R}$, U an open subset of $\mathbf{R}^{n+1}$ and the "unknown" to be found is a function $u \colon I \to \mathbf{R}$ which satisfies the above equation. The above is a differential equation of *order n*. We often work in a situation where the domain of F is not necessarily a rectangle of the form $I \times \mathbf{R}^{n+1}$ but an open set in $\mathbf{R}^{n+2}$. The exact formulation will be clarified later.

In this course we assume (1.1.1) can be written in the form

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}) \tag{1.1.2}$$

[1] See §§2.1 of Lecture 5 of ANA2.

Example 1.1.3. Here is a standard form of an ODE from more elementary courses:

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3x\frac{\mathrm{d}y}{\mathrm{d}x}+4y=0 \qquad (x>0).$$

This can be written in the form (1.1.1) by setting

$$F(a,b,c,d)=a^2d+3ac+4b$$

and noting that the differential equation then is

$$F(x,y,y',y'')=0.$$

If $a>0$, the equation $F(a,b,c,d)=0$ can be written as

$$d=f(a,b,c)$$

where

$$f(a,b,c)=\frac{1}{a^2}\Big(-3ac-4b\Big).$$

The given ODE can then be re-written in the form (1.1.2) as

$$y''=f(x,y,y'), \qquad (x>0)$$

where f is as above.

1.1.4. The dot notation. We often use the familiar "dot notation" to denote derivatives. In greater detail, suppose $\boldsymbol{\varphi}\colon I\to U$ is a differentiable map from an interval I in $\mathbf{R}$ to an open subset U of $\mathbf{R}^n$. Then

$$\dot{\boldsymbol{\varphi}}=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\varphi}.$$

1.2. **Vector first order differential equations.** A system of first order differential equations is a system of the form

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} &= v_1(t,x_1,\ldots,x_n) \\[2ex] \dfrac{\mathrm{d}x_2}{\mathrm{d}t} &= v_2(t,x_1,\ldots,x_n) \\ & \vdots \\ \dfrac{\mathrm{d}x_n}{\mathrm{d}t} &= v_n(t,x_1,\ldots,x_n) \end{cases} \tag{1.2.1}$$

where $\boldsymbol{v}=(v_1,\ldots,v_n)$ is an $\mathbf{R}^n$-valued function on some open subset Ω of $\mathbf{R}^{n+1}$. Writing $\boldsymbol{x}=(x_1,\ldots,x_n)$, this becomes an equation of the form

$$\dot{\boldsymbol{x}}=\boldsymbol{v}(t,\,\boldsymbol{x}). \tag{1.2.2}$$

Equation (1.2.2) is regarded as a *first order vector differential equation*. To ensure existence of solutions we have to put conditions on the function $\boldsymbol{v}$. A solution is by definition a function $\boldsymbol{\varphi}\colon I\to\mathbf{R}^n$ where I is an interval (usually an open interval, or else, one uses one-sided derivatives at the boundary point of I which lie in I) such that $(t,\,\boldsymbol{\varphi}(t))\in\Omega$ for $t\in I$ and $\dot{\boldsymbol{\varphi}}(t)=v(t,\boldsymbol{\varphi}(t))$ for all $t\in I$.

1.3. **Converting an n^{th} order scalar ODE into a first order vector ODE.** Consider the ODE (1.1.2), i.e. the equation $u^{(n)} = f(t, u, u', \dots, u^{(n-1)})$ where f is a function on open set Ω of $\mathbf{R}^{n+1}$. Let $\boldsymbol{w} = (w_1, \dots, w_n)\colon \Omega \to \mathbf{R}^n$ be the function given by

$$\boldsymbol{w}(t, y_1, \dots, y_n) = (y_2, \dots, y_n, f(t, y_1, \dots, y_n)).$$

Consider the first order vector DE:

$$\dot{\boldsymbol{x}} = \boldsymbol{w}(t, \boldsymbol{x}). \tag{1.3.1}$$

It is easy to see that if $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$ is a solution of (1.3.1), then φ_1 is a solution of (1.1.2). Conversely, if ψ is a solution of (1.1.2), then $\boldsymbol{\varphi} = (\psi, \dot{\psi}, \ddot{\psi}, \dots, \psi^{(n-1)})$ is a solution of (1.3.1). These two processes are inverses of each other. Note that (1.3.1) is of the form (1.1.2).

The fact that an n^{th} order scalar ODE can be converted to a first order vector ODE allows us to focus our study on the latter, i.e equations of the form (1.2.2). In fact there is a further simplification. It will turn out that it will be enough to study equations in which $\boldsymbol{v}$ does not depend on t, i.e. equations of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}). \tag{1.3.2}$$

Such differential equations are called *autonomous differential equations*

An important theorem we will prove in the course is the following.

Theorem 1.3.3. *Suppose $\boldsymbol{v}\colon \Omega \to \mathbf{R}^n$ is $\mathscr{C}^1$, where Ω is an open subset of $\mathbf{R}^{n+1}$ and suppose we have a point $(t_0, \boldsymbol{x}_0) \in \Omega$. Then the* initial value problem *(IVP)*

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x}), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

has a unique solution in a neighbourhood of t_0.

We will show that there is a "maximal interval of existence" $J_{\max} = (\omega_-, \omega_+)$ around t_0 on which the solution to the IVP exists. In other words we can find $\boldsymbol{\varphi}_{\max}\colon J_{\max} \to \Omega$ which is $\mathbf{C}^1$ such that

$$\dot{\boldsymbol{\varphi}}_{\max}(t) = \boldsymbol{v}(t, \boldsymbol{\varphi}_{\max}(t)), \quad \boldsymbol{\varphi}_{\max}(t_0) = \boldsymbol{x}_0 \qquad (t \in J_{\max})$$

and such that if I is another interval of existence for the IVP, with $\boldsymbol{\varphi}\colon I \to \Omega$ a solution to the IVP on I[2] then

- $I \subset J_{\max}$
- $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{\max}\big|_I$.

Note that $\boldsymbol{\varphi}_{\max}$ depends upon the initial data $(t_0, \boldsymbol{x}_0)$. Thus $\boldsymbol{\varphi}_{\max} = \boldsymbol{\psi}(t_0, \boldsymbol{x}_0, t)$. The time point t_0 is called the *initial time* and $\boldsymbol{x}_0$ the *initial state* or *initial phase*. In general, the $\boldsymbol{x}$'s vary in a *phase space* (also called a *phase space*), and the t's in *time space*.

1.3.4. Let U be open in $\mathbf{R}^n$ and I an open interval in $\mathbf{R}$. Let $(t_0, \boldsymbol{x}_0) \in I \times U$ and suppose $\boldsymbol{v}\colon I \times U \to \mathbf{R}^n$ is a $\mathscr{C}^1$ function. We ate generally interested in this course in the IVP as given in Theorem 1.3.3. In this case, with $\Omega = I \times U$, we can separate the time interval I (for the action represented by the IVP) from the phase space (also known as the state space) U. The set $I \times U$ is called *extended phase space*, and a solution to the IVP (or the ODE) is often called an *integral curve*

[2] This implies $t_0 \in I$, $\dot{\boldsymbol{\varphi}}(t) = \boldsymbol{v}(t, \boldsymbol{\varphi}(t))$, for $t \in I$, and $\boldsymbol{\varphi}(t_0) = \boldsymbol{x}_0$

About these notes. This lecture was given on January 4, 2021. These course notes are a reasonably faithful record of the lectures given via zoom at the Chennai Mathematical Institute (CMI) in the January-April semester of 2020-21. The course is Differential equations, a core course for B.Sc second year students at CMI. If you have comments on these notes, or on related course material, please send an email to pramath@cmi.ac.in.