DEQN COOKBOOK - II

Notations. For a non-negative integer n, we write $\mathscr{C}^n$ for $\mathscr{C}^n(\mathbf{R})$, with the understanding that $\mathscr{C}^0$ denotes the vector space of continuous functions on $\mathbf{R}$. Set

$$D = \frac{\mathrm{d}}{\mathrm{d}x}.$$

We regard D as a linear transformation on $\mathscr{C}^1$ with values in $\mathscr{C}^0$. Note that $D(\mathscr{C}^n) \subset \mathscr{C}^{n-1}$ of $\mathscr{C}^1$ for $n \geq 1$. Composing and iterating, we have a linear transformation $D^n \colon \mathscr{C}^n \to \mathscr{C}^0$, with $D^n f = \frac{\mathrm{d}^n}{\mathrm{d}x^n} f$. Note that for $D^j(\mathscr{C}^n) \subset \mathscr{C}^{n-j}$ for $n \geq j$. It follows that if $p(T) = \sum_{j=0}^{n} a_{n-j} T^j \in \mathbf{R}[T]$ is a polynomial of degree n (so that $a_0 \neq 0$), then we can form the polynomial $p(D)$ and regard it as an linear transformation on $\mathscr{C}^n$ with values in $\mathscr{C}^0$. Thus $p(D) \colon \mathscr{C}^n \to \mathscr{C}^0$. Moreover, $p(D)(\mathscr{C}^m) \subset \mathscr{C}^{m-n}$ for $m \geq n$. In explicit terms

$$p(D)f = a_0 f^{(n)} + a_1 f^{(n-1)} + \cdots + a_{n-1} f' + a_n f \qquad (f \in \mathscr{C}^n).$$

It is straightforward to verify that

$$p(D)e^{rx} = p(r)e^{rx}. \tag{1}$$

In fact, a little thought shows that (1) remains true even if $r \in \mathbf{C}$ since

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))) = (\alpha + i\beta)(e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))).$$

A *linear differential equation with constant coefficients of order* n is a differential equation of the form

$$p(D)y = g$$

where g is a continuous function on $\mathbf{R}$ and (as before) $p(T) = \sum_{j=0}^{n} a_{n-j} T^j \in \mathbf{R}[T]$ is a polynomial of degree n. In more explicit terms, this DE is

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g.$$

Such a DE is said to be (additionally) *homogeneous* if $g = 0$. In other words a homogeneous linear differential equation with constant coefficients is of the form (with p as above)

$$p(D)y = 0$$

or

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$$

The polynomial $p(T) \in \mathbf{R}[T]$ is called the *characteristic polynomial* of the DE $p(D)y = 0$.

Homogenous linear DE's with constant coefficients. Let $p(T)$ be be a real polynomial of degree n, say $p(T) = \sum_{j=0}^{n} a_{n-j}T^j$ and consider the again the homogenous DE

$$p(D)y = 0. \tag{2}$$

In other words, the solution space is exactly the null space of the linear transformation $p(D)\colon \mathscr{C}^n \to \mathscr{C}^0$. Keeping (1) in mind, we first solve the equation

$$p(t) = 0. \tag{3}$$

This is the so called *characteristic equation* or *auxiliary equation* of (2). Let r is a real root of (3) of multiplicity m, then $e^{rx}, xe^{rx}, \dots, x^{m-1}e^{rx}$ are m linearly independent solutions of (2). If $r = \alpha + i\beta$ is a non-real root of (3), i.e. $\beta \neq 0$, then its complex conjugate $\bar{r} = \alpha - i\beta$ is also a root of (3). Moreover if m is the multiplicity of the root r, then it is also the multiplicity of the root $\bar{r}$. In this case $x^j e^{\alpha x}\cos(\beta x)$, $x^j e^{\alpha x}\sin(\beta x)$, $j = 0, \dots, m-1$ give $2m$ linearly independent solutions of (2).

In general, if $r_1, \dots, r_k$ are the real roots of (3) with multiplicities $m_1, \dots, m_k$ respectively, and $\alpha_1 + i\beta_1, \alpha_1 - i\beta_1, \dots \alpha_l + i\beta_l, \alpha_i - i\beta_l$, the non-real roots of (3) written in conjugate pairs with multiplicities $\mu_1, \dots, \mu_l$ respectively (μ_j being the multiplicity of the conjugate pair $\alpha_j + i\beta_j$, $\alpha_j - i\beta_j$), then the collection $R \cup C \cup S$ forms a basis for the vector space of solutions of (2), i.e. of the null space of $p(D)$, where

$$\begin{aligned}
R &= \bigcup_{i=1}^{k} \Big\{ e^{r_i x}, xe^{r_i x}, \dots, x^{m_i - 1}e^{r_i x} \Big\} \\
C &= \bigcup_{j=1}^{l} \Big\{ e^{\alpha_j x}\cos(\beta_j x), xe^{\alpha_j x}\cos(\beta_j x), \dots, x^{\mu_j - 1}e^{\alpha_j x}\cos(\beta_j x) \Big\} \\
S &= \bigcup_{j=1}^{l} \Big\{ e^{\alpha_j x}\sin(\beta_j x), xe^{\alpha_j x}\sin(\beta_j x), \dots, x^{\mu_j - 1}e^{\alpha_j x}\sin(\beta_j x) \Big\}.
\end{aligned}$$

The proofs of the statements will take a few lectures in class. Note that

$$k + 2\sum_{j=1}^{l} \mu_j = n.$$

As a (somewhat extreme) example, consider the eighth order DE:

$$y^{(8)} + 4y^{(7)} + 7y^{(6)} + 6y^{(5)} - 6y^{(3)} - 7y'' - 4y' - y = 0.$$

The characteristic equation is

$$r^8 + 4r^7 + 7r^6 + 6r^5 - 6r^3 - 7r^2 - 4r - 1 = 0.$$

If we denote the characteristic polynomial by p, then one checks that the factorisation of p into a product of irreducible polynomials over $\mathbf{R}$ is:

$$p(r) = r^8 + 4r^7 + 7r^6 + 6r^5 - 6r^3 - 7r^2 - 4r - 1 = (r-1)(r+1)^3(r^2+r+1)^2.$$

The complex roots then are $r = 1$ with multiplicity 1, $r = -1$ with multiplicity 3, $r = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ with multiplicity 2, and $r = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ with multiplicity 2. According to the recipe we have given, the general solution of $p(D)y = 0$ is

$$y = c_1 e^x + (c_2 + c_3 x + c_4 x^2)e^{-x} + (c_5 + c_6 x)e^{-(1/2)x}\cos(\tfrac{\sqrt{3}}{2}x) \\ + (c_7 + c_8 x)e^{-(1/2)x}\sin(\tfrac{\sqrt{3}}{2}x)$$

where the c_i, $i = 1, \dots, 8$, are arbitrary constants.

The method of undetermined coefficients. Consider a linear DE

$$p(D)y = g \tag{4}$$

where $p \in \mathbf{R}[T]$ is a polynomial of degree n and g is a function of the form

$$g(x) = e^{\alpha x}\Big(P_a(x)\cos(\beta x) + Q_b(x)\sin(\beta x)\Big)$$

where P_a is a polynomial of degree a and Q_b a polynomial of degree b. Let $d = \max(a, b)$. Note, it is possible for β to equal zero, in which case $g(x) = e^{\alpha x}P_a(x)$. If $r = \alpha + i\beta$ is not a root of the characteristic polynomial, then a particular solution is of the form

$$y_p = e^{\alpha x}\Big(A(x)\cos(\beta x) + B(x)\sin(\beta x)\Big)$$

where $A$and B are polynomials of degree d. More generally, if we regard a non-root of p as a root with multiplicity zero, then the general form of a particular solution is

$$y_p = x^s e^{\alpha x}\Big(A(x)\cos(\beta x) + B(x)\sin(\beta x)\Big)$$

where s is the multiplicity of $r = \alpha + i\beta$ as a root the characteristic polynomial, and A and B are polynomials of degree d.

If $s \neq 0$, i.e. if $r = \alpha + i\beta$ is a root of p, then the equation is a classified as a *resonance case*. If r is not a root of p, then we are in the *non-resonance case*. The coefficients of A and B can be determined by considering the identity

$$p(D)y_p = e^{\alpha x}\Big(P_a(x)\cos(\beta x) + Q_b(x)\sin(\beta x)\Big)$$

and comparing the coefficients of $\cos(\beta x)$ and $\sin(\beta x)$ on both sides (after cancelling the nowhere vanishing function $e^{\alpha x}$ from both sides).

The principle of superposition. Consider the DE $p(D)y = g$, where as before $p(T)$ is a polynomial of degree n (or more generally a "linear differential operator") and g a continuous function on $\mathbf{R}$. Suppose $g = g_1 + g_2$, where g_1 and g_2 are continuous functions on $\mathbf{R}$. If y_i is a solution of $p(D)y = g_i$, $i = 1, 2$, then $y_1 + y_2$ is clearly a solution of $p(D)y = g$. This obvious observation is called *the principle of superposition.*

Complementary solutions and the particular solutions. In the DE $p(D)y = g$ discussed above, the general solution of the *associated homogeneous DE*, namely, $p(D)y = 0$, is called the *complementary solution* to the DE. It is not a solution to the DE (unless $g \equiv 0$), and is denoted y_c. Any solution to the given DE is called a *particular solution* and once one picks one, it is usually denoted y_p (where the subscript p has nothing to do with the polynomial p). The general solution to the given DE is then $y = y_c + y_p$.

Here are three examples

(a) $y''' + 5y'' - 14y' = x - 3$.

Solution: The equation can be re-written as $p(D)y = x - 3$ where $p(t) = t^3 + 5t^2 - 14t$. The roots are $t = 0, 2, 7$, each with multiplicity one. The right side can be re-written as $e^{0\cdot x}(x-3)\cos(0\cdot x)$ and so this is really a resonance case. Since the multiplicity of the root 0 is one, the form of the solution is $y_p = x(Ax + B)$. Now $y_p' = 2Ax + B$, $y_p'' = 2A$ and $y''' = 0$. Substituting y_p for y in the given DE, we get

$$0 + 5(2A) - 14(2Ax + B) = x - 3.$$

This gives

$$-28Ax + 10A - 14B = x - 3.$$

Comparing coefficients, we see that $-28A = 1$ and $10A - 14B = -3$. Solving we get $A = -\frac{1}{28}$ and $B = \frac{37}{196}$. Thus

$$y_p = \tfrac{1}{28}x + \tfrac{37}{196}$$

is a particular solution. Clearly the complementary solution is $y_c = c_1 + c_2e^{2x} + c_3e^{7x}$ where c_1, c_2, c_3 are arbitrary constants. The general solution is therefore

$$y = c_1 + c_2e^{2x} + c_3e^{7x} + \tfrac{1}{28}x + \tfrac{37}{196},$$

where c_1, c_2, c_3 are arbitrary constants.

(b) $y^{(4)} - y = 2\cos x$.

Solution: The equation can be re-written as $p(D)y = x - 3$ where $p(t) = t^4 - 1$. Now $p(t) = (t^2 - 1)(t^2 + 1)$ and hence the roots of characteristic equation are $t = \pm 1, \pm i$. This is a resonance case again, for $i = \alpha + i\beta$ with $\alpha = 0$ and $\beta = 1$, and hence $2\cos x = 2e^{\alpha x}\cos(\beta x)$. So the form of the particular solution is $y_p = x(A\cos x + B\sin x)$. Substituting y_p for y in the given DE and comparing coefficients, one finds that $A = 0$ and$B = -\frac{1}{2}$. The exact computations are left to you (please do the calculations). Thus $y_p = -\frac{1}{2}x\sin x$. Note that $y_c = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x$ where the c_i's are arbitrary constant. Therefore the general solution is

$$y = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x - \frac{1}{2}x\sin x$$

where the c_i's are arbitrary constants.

(c) Write down the form of a particular solution to

$$y^{(8)} + 4y^{(7)} + 7y^{(6)} + 6y^{(5)} - 6y^{(3)} - 7y'' - 4y' - y$$
$$= 3e^{2x} - 5\sin x - 5xe^{-x} + 2x^2e^{-(1/2)x}\sin(\tfrac{\sqrt{3}}{2}x).$$

Solution: We have already seen that the roots of the characteristic polynomial of the associated homogeneous linear DE are $1, -1, -1, -1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, where we have listed each root as many times as the multiplicity with which it occurs. Check (from the recipe given) that y_p has the form

$$y_p = Ae^{2x} + (B\cos x + C\sin x) + x^3(Dx + E)e^{-x}$$
$$+ x^2e^{-(1/2)x}\Big\{(Fx^2 + Gx + H)\cos\Big(\tfrac{\sqrt{3}}{2}x\Big) + (Ix^2 + Jx + K)\sin\Big(\tfrac{\sqrt{3}}{2}x\Big)\Big\}.$$

Here, $A, B, C, D, E, F, G, H, I, J, K$ are constants. Working out the constants is very tedious (though straightforward, if you have time) task, which thankfully we have not been asked to do.

Sporadic techniques. If an equation is of the form

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha x \frac{\mathrm{d}y}{\mathrm{d}x} + \beta y = 0, \quad x > 0, \tag{5}$$

with α and β constants, then the substitution $t = \ln x$ transforms the equation to:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (\alpha - 1)\frac{\mathrm{d}y}{\mathrm{d}t} + \beta y = 0. \tag{6}$$

This is a linear second order homogeneous DE with constant coefficients and we can solve for y as a function of t. Substituting $t = \ln x$, we get the expression for y in terms of x. Differential Equations of the form (5) are called *Euler's equations.*

More generally, if we have an equation of the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0, \tag{7}$$

on an interval I, with $q > 0$ on I, and if

$$\frac{q'(x) + 2p(x)q(x)}{2(q(x))^{3/2}}$$

is a constant on I, then the transformaton

$$t = \int \sqrt{q(x)}dx \tag{8}$$

transforms (7) into an equation with constant coefficients.

Problems. Solve (if no initial conditions are given, find the general solution):

1) $2y'' + 3y' - 2y = 0$

2) $9\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 24\frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 0$

3) $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 6\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 10y = 0$, $y(0) = 2$, $y'(0) = -7$ and $y''(0) = 47$

4) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} - 7y = 0$

5) $2x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0$

6) $9x^2y'' + 33xy' + 16y = 0$

7) $x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} - 7y = 0$

8) $x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (x^2 - 1)\frac{\mathrm{d}y}{\mathrm{d}x} + x^3 y = 0, \quad x > 0.$

9) $y^{(6)} - 3y^{(5)} + 40y^{(3)} - 180y'' + 324y' - 432y = 0$. [Hint: $1 + i\sqrt{5}$ is a root (with positive multiplicity) of the characteristic polynomial.]

10) $4y'' + 11y' + 6y = 0$, $y(0) = 5$, $y'(0) = -5$

Find a particular solution of

11) $3y'' - 2y' - 2y = \cos(2x)$

12) $2y'' + 7y' + 6 = e^{-2x}$

13) $2x^2 \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5x \dfrac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^2 - 3\sqrt{x}$

14) $x^2 \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x \dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sin(\ln x) + x^2$

15) $y'' - 2y' + 17 = (3x^2 + 2)e^x \sin(2x)$

16) $x \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (x^2 - 1) \dfrac{\mathrm{d}y}{\mathrm{d}x} + x^3 y = \exp\left(-\tfrac{1}{4}x^2\right) \cos\left(\tfrac{\sqrt{3}}{4}x^2\right), \quad x > 0.$

Write down the form of a particular solution of

17) $y^{(4)} - 4y^{(3)} + 38y'' - 68y + 289 = xe^x \cos(2x)$

18) $y^{(6)} - 3y^{(5)} + 45y^{(4)} - 24y^{(3)} + 236y'' + 1300y' - 4056y = x^2 e^x \cos(\sqrt{5}x)$

The next two questions involve linear differential equations with non-constant coefficients - these are equations you have encountered in some of the problems above. The recipe for solving these were given in the main part of the text.

19) Show that the change of variables $t = \ln x$ transforms the DE in equation (5) of the text portion of this document to the one in (6).

20) Show that the change of variables $t = \int q(x)^{1/2} dx$ in (8) (of the text portion of this document) transforms the DE in equation (7) of the text to a linear DE with constant coefficients, provided q is positive everywhere on the interval of interest and the expression $(q'(x) + 2p(x)q(x))/(2(q(x))^{3/2})$ is a constant.