DEQN COOKBOOK - I

Separable DE's. These are equations of the form

$$g(y)\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

We assume f and g are continuous. If G is a primitive (i.e. anti-derivative) of g, and F a primitive of f, then upon integrating the above equation with respect to x, we get

$$G(y) = F(x) + C$$

where C is an arbitrary constant. This defines y as an implicit function of x. An example would be

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\sin(2x)}{3-2y}$$

which can be re-written as $(3-2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sin(2x)$. A general solution is $3y - y^2 = -\cos(2x) + C$ with C, as always, an arbitrary constant. If an "initial value" of y is prescribed, say $y(x_0) = y_0$, then C can be worked out. In fact $C = G(y_0) - F(x_0)$. Note that for the implicit function theorem to apply (so that y is a function of x in an open neighbourhood of x_0), we need $\frac{\partial\Phi}{\partial y}(x_0,\, y_0)$ to be *non-zero*, where $\Phi(x,y) = G(y) - F(x)$. This is the same as the condition that $g(y_0) \neq 0$. In the example we have give, this means $y_0 \neq \frac{3}{2}$.

Linear First Order DE's. These are of the form

$$y' + p(x)y = q(x)$$

where p and q are continuous functions on an interval. Recall from your high-school that the trick is to multiply the equation by an integrating factor $\mu(x)$ given by $\mu(x) = \exp\int p(x)dx$. The equation then is equivalent to $\frac{\mathrm{d}\mu(x)y}{\mathrm{d}x} = \mu(x)q(x)$, whence $y = \mu(x)^{-1}\int \mu(x)q(x)dx$. Note that if $P(x)$ is a primitive of $p(x)$, then so is $P_1(x) = P(x) + D$ where D is a constant. If $\mu_1(x) = \exp(P_1(x))$, then $\mu_1(x) = e^D\mu(x)$, whence $\mu_1(x)^{-1}\int \mu_1(x)q(x)dx = \mu(x)^{-1}e^{-D}\int e^D\mu(x)q(x)dx = \mu(x)^{-1}\int \mu(x)q(x)dx$.

Homogeneous first order DE's. The term *homogeneous is overused!* There are at least two different kinds of DE's called homogeneous. In the course you will also learn about homogeneous linear differential equations which have nothing to do with the homogeneous DE's we now discuss (which you also saw in your high school). An expression $f(x,y)$ in two variables is called *homogeneous of degree zero* if $f(\lambda x, \lambda y) = f(x,y)$ for every $\lambda \neq 0$. Homogenous first order equations are equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$$

where f is homogeneous of degree 0. Recall (from high school) that such equations are solved by setting $y = vx$ (with x assumed non-zero in the interval of interest).

Then, if we set $g(v) = f(1, v)$, and use the identity $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$, the equation reduces to

$$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = g(v)$$

which is separable (check!). As an example, consider, for $x > e^{-3}$,

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = xy + y^2, \quad y(1) = -\tfrac{1}{3}.$$

The main equation (without the additional conditions) can be re-written in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy+y^2}{x^2}$ which is homogeneous. From the discussion above, setting $v = y/x$, this reduces to $x\frac{\mathrm{d}v}{\mathrm{d}x} + v = v + v^2$, i.e. to

$$v^{-2}\frac{\mathrm{d}v}{\mathrm{d}x} = x^{-1}.$$

Check that on solving this separable equation and substituting $y = vx$, this yields $y = \frac{-x}{\ln x + C}$ where C is an arbitrary constant. Since $y(1) = -\frac{1}{3}$, this means $C = 3$, i.e.

$$y = \frac{-x}{\ln x + 3} \qquad (x > e^{-3}).$$

Bernoulli Equations. These are equations of the form

$$y' + p(x)y = q(x)y^n$$

with $n \in \mathbf{R}$, where p and q are continuous functions on some interval. If $n = 0$ this is simply a first order linear DE. If $n = 1$ this is a separable DE. For $n \neq 0, 1$, the substitution $v = y^{1-n}$ gives us a first order linear DE in v, namely (check this):

$$\tfrac{1}{1-n}v' + p(x)v = q(x).$$

Setting $y = v^{1/(1-n)}$, we get a solution. If $n > 0$, there is another hidden solution, namely $y \equiv 0$.

Exact DE's. Consider first order DE's of the form

$$(*) \qquad M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

with M and N being $\mathscr{C}^1$ functions on a region U in $\mathbf{R}^2$. The DE $(*)$ is said to be *exact* if there exists a $\mathscr{C}^2$ function P on U such that

$$(**) \qquad \frac{\partial P}{\partial x} = M \quad \text{and} \quad \frac{\partial P}{\partial y} = N.$$

Then $(*)$ amounts to saying $\frac{\mathrm{d}P(x,y(x))}{\mathrm{d}x} = 0$ where $y(x)$ is a solution of $(*)$. Equivalently, y is given by the implicit equation

$$P(x, y) = C.$$

Note that if P a function satisfying $(**)$ then it is $\mathscr{C}^2$ and hence $\frac{\partial^2 P}{\partial y \partial x} = \frac{\partial^2 P}{\partial x \partial y}$. This amounts to saying

$$(\dagger) \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Conversely, assuming $U = \mathbf{R}^2$ (or more generally U is "simply connected"), if (†) is satisfied, then by Green's theorem we have, for any closed simple curve γ in $\mathbf{R}^2$ (or in U, with U simply connected) enclosing a region R, we have:

$$\oint_\gamma (M\mathrm{d}x + N\mathrm{d}y) = \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mathrm{d}x\mathrm{d}y = 0.$$

In other words $\int (M\mathrm{d}x + N\mathrm{d}y)$ is path independent. Fix a "base point" (x_0, y_0) in $\mathbf{R}^2$. For a variable point (x, y) in $\mathbf{R}^2$, the symbol $\int_{(x_0,y_0)}^{(x,y)} (M\mathrm{d}x+N\mathrm{d}y)$ makes sense by the just established path independence . Define

$$P(x, y) = \int_{(x_0,y_0)}^{(x,y)} (M\mathrm{d}x + N\mathrm{d}y).$$

It is easy to see that P satisfies ($**$). Thus on $\mathbf{R}^2$ (or on a simply connected region U), the equation ($*$) is exact if and only if (†) is satisfied.

We point out that if we have another function satisfying ($**$), say Q such that $Q_x = M$ and $Q_y = N$, then the partial derivatives of $P - Q$ are zero, and hence $P-Q$ is a constant. Hence the family of solutions $Q(x, y) = D$, where we have one solution for each arbitrary constant D, is no different from the family of solutions given by $P(x, y) = C$, with C an arbitrary constant.

As an example, consider the DE

$$3x^2y + y\cos(xy) + (x^3 + x\cos(xy) + y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Setting $M(x, y) = 3x^2y + y\cos(xy)$ and $N(x, y) = x^3 + x\cos(xy) + y^2$ we see that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3x^2 + \cos(xy) - xy\sin(xy).$$

Thus the given DE is exact. We need to find P such that $P_x = M$ and $P_y = N$.

$$P(x, y) = \int M(x, y)\mathrm{d}x = x^3y + \sin(xy) + g(y),$$

where g is an arbitrary differentiable function of y. Differentiating the above expression with respect to y we get

$$N(x, y) = P_y(x, y) = x^3 + x\sin(xy) + g'(y).$$

Comparing this with the given expression for N, we get $g'(y) = y^2$, whence we can pick $g(y)$ to be $\frac{1}{3}y^3$. With this choice of g, we have $P(x, y) = x^3y + \sin(xy) + \frac{1}{3}y^3$, whence y is defined by the implicit equation

$$x^3y + \sin(xy) + \tfrac{1}{3}y^3 = C$$

where C is a constant. Some additional observations are worth making.

- The constant C can be worked out based on additional data, say $y(x_0) = y_0$ where x_0 and y_0 are known.
- For x_0 and y_0 as above, we need to make sure that $P_y(x_0, y_0) \neq 0$, i.e. $N(x_0, y_0) \neq 0$, for the implicit function theorem to apply.
- The two comments above apply to general exact DE's, not just to the example worked out.

Second order equations with the dependent variable missing. The most general form of a second order DE we are interested in is of of the form $y'' = f(x, y, y')$. Consider a DE of the form

$$y'' = f(x, y').$$

Note that the dependent variable y is missing. In this case, setting $v = y'$, the above equation becomes $v' = f(x, v)$, which is a first order equation in v. If we solve this, we can find y by integrating v with respect to x. The process will introduce two arbitrary constants, and one can solve for the constants if $y(x_0)$ and $y'(x_0)$ are given for a fixed value x_0 of x.

As an example, consider the equation

$$y'' + x(y')^3 = 0 \qquad (x > 0,\, y(1) = 0,\, y'(1) = 1).$$

This translates to $v' + xv^3 = 0$, which is separable. We have $v^{-3}v' = -x$, i.e. $v^{-2} = x^2 + D$, where D is a constant. Since $v(1) = y'(1) = 1$, we get $D = 0$. Thus $v^{-2} = x^2$. Since $v(1) > 0$, we must have (by continuity of v), $v = x^{-1}$ for $x > 0$. This gives $y = \ln(x) + E$. (Note: We have been given that $x > 0$.) Using the fact that $y(1) = 0$, we get $y = \ln x$. One observation. If we look at the DE $v' + xv^3 = 0$ without worrying about the initial values, there is an obvious solution, namely $v \equiv 0$. In other words, y is a constant. But in our case, we discard that solution because we have $v(1) = y'(1) = 1$, which is not possible if $v \equiv 0$. If the initial conditions are not given, then in the general solution, we have to consider this hidden solution too.

Second order equation with the independent variable missing. These are equations of the type

$$y'' = f(y, y').$$

Here we once again set $v = y'$, but write $y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}y}$. Thus the DE is transformed to:

$$v\frac{\mathrm{d}v}{\mathrm{d}y} = f(y, v).$$

One solves for v as a function of y, (introducing an arbitrary constant), and then one solves the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = v(y)$, which is separable. The last step introduces a second arbitrary constant.

An example is the equation (used to compute "escape velocity"):

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{gR^2}{(R+y)^2}$$

where y is the height above the ground, R radius of the earth, and g the acceleration due to gravity at sea level. The independent variable t marks time. Setting $v = \frac{\mathrm{d}y}{\mathrm{d}t}$ (v is the velocity in the upward direction), we get, from the above discussion

$$v\frac{\mathrm{d}v}{\mathrm{d}y} = -\frac{gR^2}{(R+y)^2}.$$

This in turn yields

$$\frac{v^2}{2} = \frac{gR^2}{R+y} + c.$$

Assume $y = 0$ when $t = 0$. If v_0 is the initial velocity then $c = \frac{1}{2}v_0^2 - gR$. This gives, for a rising body (so that v and y are both pointed upwards)

$$v = \sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+y}}.$$

If ζ is the maximum height reached, then $v = 0$ when $y = \zeta$, whence we have the rlation $v_0^2 - 2gR + 2gR^2/(R+\zeta) = 0$. This gives

$$v_0 = \sqrt{2gR\frac{\zeta}{R+\zeta}}.$$

This is to be interpreted as the formula for the initial velocity if the maximum altitude to be reached is ζ. If we let ζ approach infinity, we get the escape velocity v_e, the initial velocity needed for a body to escape earth's gravitational pull. Clearly (by letting $\zeta \to \infty$ in the expression for v_0 as a function of ζ) we have

$$v_e = \sqrt{2gR}.$$

Problems. Solve

1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1+y^2}{x}, \quad x > 0$

2) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2+xy+y^2}{x^2}$

3) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3x^2-2x-1}{2(y-1)}$, $y(3) = 1 - \sqrt{13}$

4) $(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 1 + x, \quad x > 0$

5) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x-2x^3}{16+2y^3}$

6) Let $y = \varphi(x)$ be a solution of the problem in **5)** such that $\varphi(x_0) = y_0$, and such that φ is defined as a $\mathscr{C}^1$ function in a neighbourhood of x_0. What are the forbidden values of y_0?

7) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x-e^{-x}}{y+e^y}$

8) $x^2y' + 2xy - y^3 = 0, \quad x > 0$

9) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^2-4x^2}{2xy}$

10) $t\ln t\dfrac{\mathrm{d}u}{\mathrm{d}t} + u = te^t, \quad t > 1$

11) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = e^{-2x}$

12) $\sin x\dfrac{\mathrm{d}y}{\mathrm{d}x} + (\cos x)y = e^x$

13) $(e^x \sin y - 2y\sin x)dx + (e^x \cos y + 2\cos x)dy = 0$

14) $(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 1 + x, \quad x > 0$

15) $xy' = y + x^2 \sin x, \quad y(\pi) = 0$

16) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{2}{x}y = \dfrac{y^3}{x^2}$

17) $y\cos x + 2xe^y + (\sin x + x^2 e^y - 1)y' = 0$

18) $x^2 y'' + 2xy' - 1 = 0, \;\; x > 0$

19) $xy' + y = \sqrt{x}, \;\; x > 0$

20) $xy'' + y' = 1, \quad x > 0$

21) $yy'' - (y')^3 = 0$

22) $2y^2 y'' + 2y(y')^2 = 1$

In problems **23)** and **24)** below:

(a) Find the solution of the initial value problem in explicit form.
(b) Determine the interval in which the solution is defined.

23) $y' = (1-2x)y^2, \; y(0) = -\frac{1}{6}$

24) $y^2\sqrt{1-x^2}\,y' = \arcsin x, \; y(0) = 1$

25) Check if the DE $xye^{x^2 y} + x^2 e^{x^2 y} y' = 0$ is exact and solve it if it is.

26) Check if the DE

$$\Big(3x^2 y \sin(x+y) + x^3 y \cos(x+y) + y \sec^2(xy)\Big)dx + \Big(x^3 \sin(x+y) + x^3 y \cos(x+y) + x \sec^2(xy)\Big)dy = 0$$

is exact and solve it if it is.

In problems **27)** and **28)**, find the value of α which makes the given DE exact.

27) $\alpha y e^{2xy} dx + (xe^{2xy} + y)dy = 0$

28) $(x+y)y^2 dx + (x^2 y + \alpha x y^2)dy = 0$