HW 9

Due date: April 18, 2021

Cookbook problems.

1) Problem 23 of Cookbook-I.

Notations. Let us re-write the DE in Problem (**1**) in terms of time and phase. In other words as:

$$(\Delta) \qquad \dot{x} = (1-2t)x^2.$$

The extended phase space Ω is clearly $\mathbf{R}^2$ since the function $v(t,\, x) = (1-2t)x^2$ is defined on all of $\mathbf{R}^2$. For $(\tau, a) \in \mathbf{R}^2$ let $(\Delta)_{(\tau,a)}$ be as usual the IVP:

$$(\Delta)_{(\tau,\, a)} \qquad \dot{x} = (1-2t)x^2, \qquad x(\tau) = a.$$

As usual, we will denote the solution of $(\Delta)_{(\tau,\, a)}$ by $\varphi_{(\tau,\, a)}$ and its maximal interval of existence by $J(\tau,\, a)$. For example, Problem (**1**) is asking you to find $\varphi_{(0,\, -\frac{1}{6})}$ and $J(0,\, -\frac{1}{6})$.

2) Let $(\tau,\, a) \in \mathbf{R}^2$. Find a formula for $\varphi_{(\tau,a)}(t)$ for $t \in J(\tau,\, a)$. You do not have to calculate $J(\tau,\, a)$ (for this problem). **Hint:** First assume $a \neq 0$ and find a formula for $\varphi_{(\tau,a)}(t)$, and then see that the formula works for $a = 0$ too.

Regions in the $(\tau,\, a)$**-plane.** Consider the diagram in the next page on the (τ, a)-plane. The regions labelled 1, 2, 2', and 3 are open regions. Thus the τ-axis is not in either region 1 or in region 3. Similarly the disconnected curve C whose equation is $a(2\tau-1)^2 = 4$, i.e. the red curve, is not in regions 1, 2, or 2'. The left branch of C borders 1 and 2', while the right branch borders 1 and 2. In what follows, for $(\tau, a) \in \mathbf{R}^2$ we set

$$\delta = \delta(\tau,\, a) := a^2(2\tau-1)^2 - 4a.$$

3) Show that if $(\tau,\, a)$ is on the τ-axis then $J(\tau, a) = \mathbf{R}$.

4) Show that if $(\tau,\, a)$ is in region 1 then $J(\tau, a) = \mathbf{R}$.

5) Show that if $(\tau,\, a)$ is in region 2, 2' or 3, then $\delta(\tau, a) > 0$.

6) Show that if $(\tau,\, a)$ is in region 2 then $J(\tau, a) = \left(\frac{1}{2} + \frac{\sqrt{\delta(\tau,\, a)}}{2|a|}, \infty\right)$.

7) Show that if $(\tau,\, a)$ is in region 2' then $J(\tau, a) = \left(-\infty, \frac{1}{2} - \frac{\sqrt{\delta(\tau,\, a)}}{2|a|}\right)$.

8) Show that if $(\tau,\, a)$ is in region 3 then $J(\tau, a) = \left(\frac{1}{2} - \frac{\sqrt{\delta(\tau,a)}}{2|a|}, \frac{1}{2} + \frac{\sqrt{\delta(\tau,a)}}{2|a|}\right)$.

9) Show that if (τ, a) is on the right branch of the curve $a(2\tau-1)^2 = 4$, then $J(\tau, a) = (\frac{1}{2}, \infty)$.

10) Show that if (τ, a) is on the left branch of the curve $a(2\tau-1)^2 = 4$, then $J(\tau, a) = (-\infty, \frac{1}{2})$.

The set $\widetilde{\Omega}$. Let S, F_1, and F_2 be the closed sets:

$$S = \{(t,\, \tau,\, a) \in \mathbf{R}^3 \mid at^2 - at - a\tau^2 + a\tau + 1 = 0\}$$

$$F_1 = \left\{(t,\, \tau,\, a) \in \mathbf{R}^3 \mid (\tau, a) \text{ in the closure of } ② \text{ and } t \le \tfrac{a+\sqrt{\delta(\tau,a)}}{2|a|}\right\}$$

$$F_2 = \left\{(t,\, \tau,\, a) \in \mathbf{R}^3 \mid (\tau, a) \text{ in the closure of } ②' \text{ and } t \ge \tfrac{a-\sqrt{\delta(\tau,a)}}{2|a|}\right\}.$$

Let F be the closed set

$$F = S \cup F_1 \cup F_2.$$

11) Let $\widetilde{\Omega} \subset \mathbf{R} \times \mathbf{R}^2$ be as in [Lecture 23, (2.1.1)]. According to Proposition 2.1.4 of *loc.cit.* it is an open subset of $\mathbf{R}^3$. Let W be the complement in $\mathbf{R}^3$ of the closed set F above. Show that $\widetilde{\Omega}$ is the connected component of W containing the origin $\mathbf{0}$. **Remark:** If you have done Problem (**2**) correctly, you will notice that $(t,\, \tau,\, a) \mapsto \varphi_{(\tau,\, a)}(t)$ is $\mathscr{C}^\infty$ on $\widetilde{\Omega}$. You don't have to prove this.

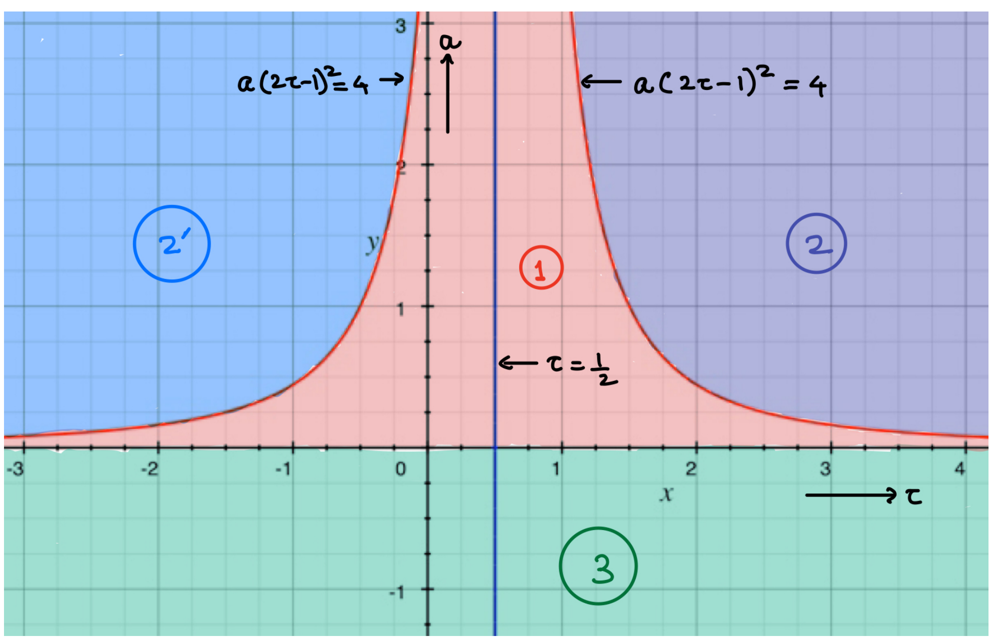

FIGURE 1.

Since you worked so hard, as a reward here are some pictures to help you visualise the answers you have got. The graphs are in the $(t,\, \tau,\, a)$ space, i.e. in $\mathbf{R}^3$. The coloured lines in space are parallel to the t-axis, and the a-axis is the vertical axis. The coloured lines are segments in which t varies for fixed (τ, a) and represent $J(\tau,\, a) \times \{\tau\} \times \{a\}$. The colours of the line segment correspond to the colour of the region that (τ, a) belongs to in FIGURE 1. The surface is the surface S above, i.e. the locus where the polynomial $f(t,\, \tau,\, a) = at^2 - at - a\tau^2 + a\tau + 1$ vanishes. It is a disconnected surface (over $\mathbf{R}$).

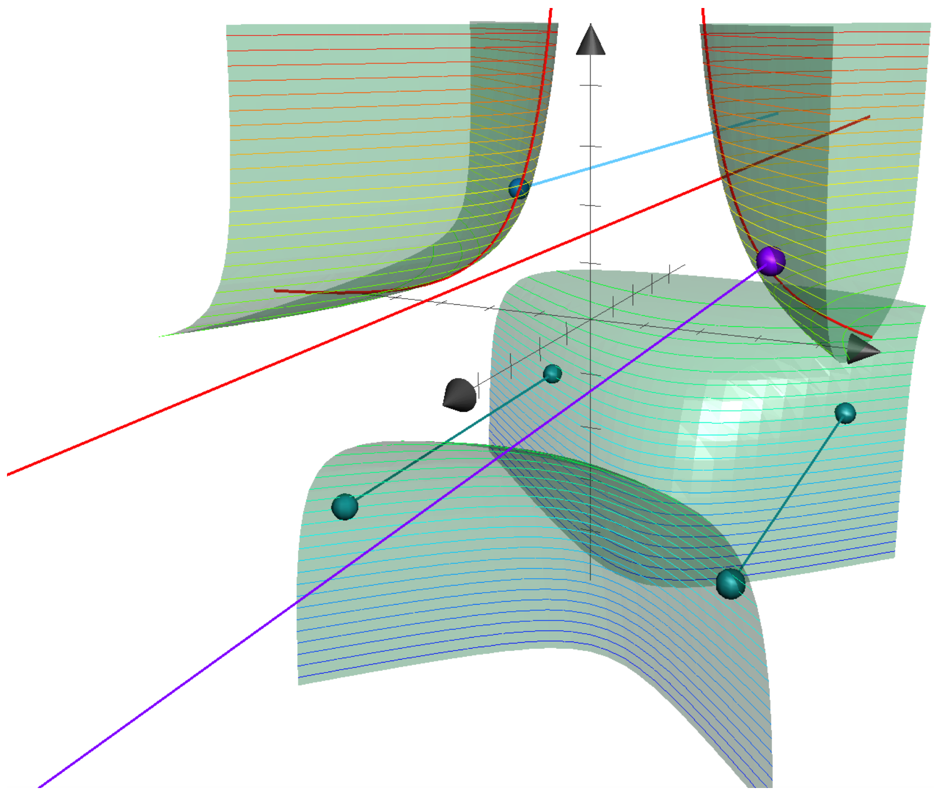

FIGURE 2. The red line is $J(\boldsymbol{\xi}) \times \{\boldsymbol{\xi}\}$ for a point $\boldsymbol{\xi}$ in the red region of FIGURE 1. Notice it encounters no barriers from S. On the other hand the other line segments do encounter barriers. The blue, purple and green line segments have similar descriptions. The solid dots and not part of the open line segments $J(\boldsymbol{\xi}) \times \{\boldsymbol{\xi}\}$. The red curves on S are the left and right branches of $a(2\tau - 1)^2 = 4$ in the plane $t = \frac{1}{2}$.

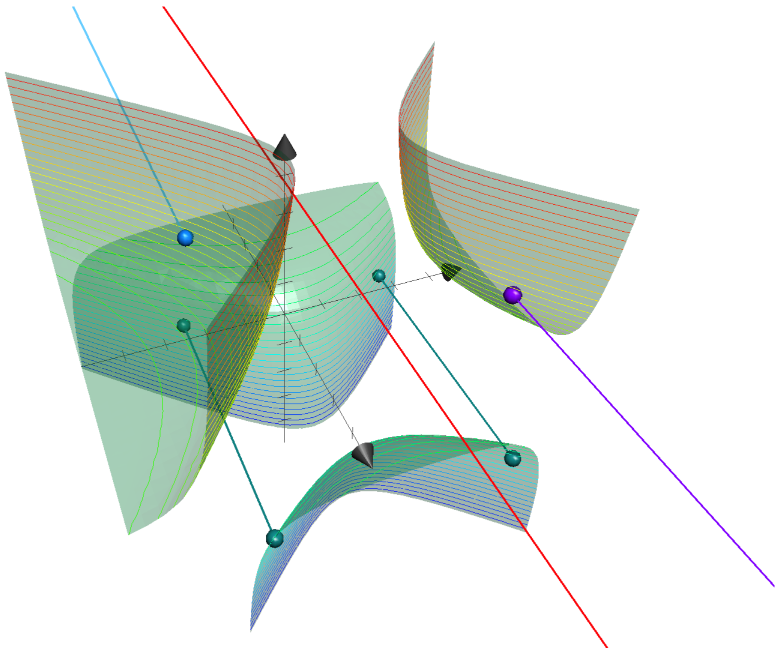

FIGURE 3. Another view.

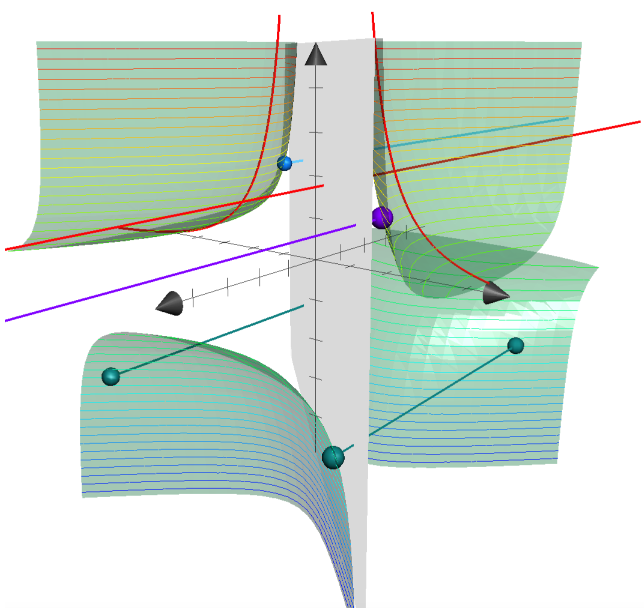

FIGURE 4. The grey plane in the picture is the plane $t = \tau$. Notice it intersects every one of the coloured line segments but does not intersect S. What is the significance of this observation?

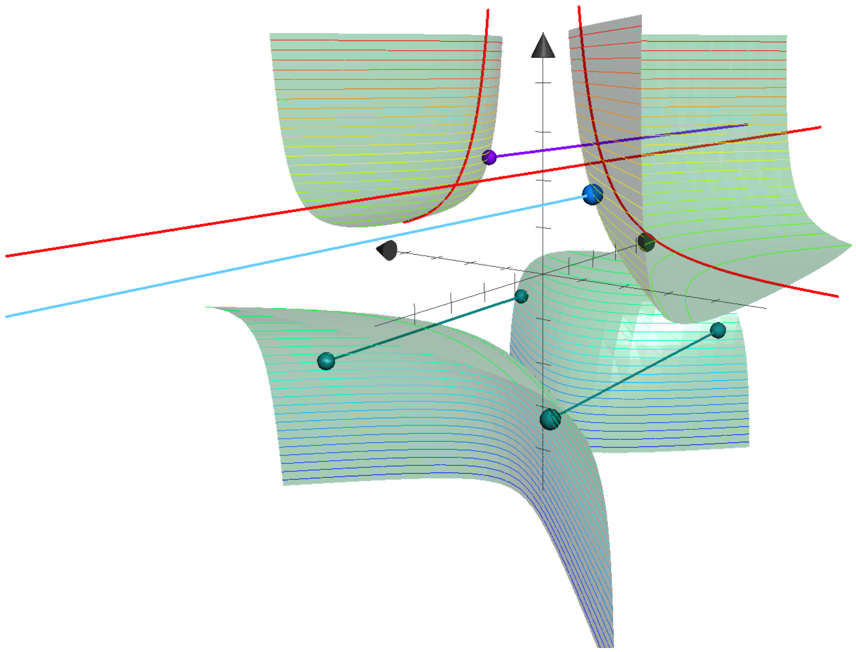

FIGURE 5. View from the 7^{th} octant, i.e. the octant where the first two co-ordinates are negative and the last one positive. Notice in this picture the blue line segment is closer to the viewer than the purple line segment. Note the directions of the positive t and τ axes.