HW 8

Due date: March 30, 2021

Let $\boldsymbol{\vartheta}$ be the standard vector field on $\mathbf{R}$, namely the one given as a derivation by the usual differentiation of functions on $\mathbf{R}$. Note that $\boldsymbol{\vartheta}$ is nowhere vanishing, and hence every smooth vector field on $\mathbf{R}$ or an open subset of $\mathbf{R}$ is of the form $f\boldsymbol{\vartheta}$ for a smooth function f. This is because the tangent space $T_y(\mathbf{R})$ is one dimensional at every $y \in \mathbf{R}$, and hence $\boldsymbol{\vartheta}(y)$ is necessarily a basis vector of $T_y(\mathbf{R})$.

Let $\boldsymbol{S}^1$ be the unit circle in $\mathbf{R}^2 = \mathbf{C}$ and let $e\colon \mathbf{R} \to \boldsymbol{S}^1$ be the Euler map $\theta \mapsto e^{i\theta}$. There is a canonical nowhere vanishing vector field on $\boldsymbol{S}^1$, namely the one which at $e(\theta_0)$ assigns the unit tangent vector $\boldsymbol{v}(e(\theta_0)) = -\sin(\theta_0)\mathbf{i} + \cos(\theta_0)\mathbf{j}$. It is easy to see that this is well defined, i.e. if $e(\theta_0) = e(\theta_1)$, then $\boldsymbol{v}(\theta_0) = \boldsymbol{v}(\theta_1)$. In terms of derivations, $\boldsymbol{v}(e(\theta_0))$ is the derivation

$$f \mapsto \frac{d(f \circ e)}{d\theta}\bigg|_{\theta=\theta_0}.$$

where f is the germ of a smooth function in a neighbourhood of $e(\theta_0)$. This vector field $\boldsymbol{v}$ on $\boldsymbol{S}^1$ is universally denoted $\frac{d}{d\theta}$. Since $\frac{d}{d\theta}$ never vanishes, every vector field on $\boldsymbol{S}^1$ can be written uniquely as $f\frac{d}{d\theta}$.

We now give two well known charts of $\boldsymbol{S}^{\mathbf{1}}$ which form an atlas. Let $U_{-1} = \boldsymbol{S}^1 \smallsetminus \{-1\}$ and $U_1 = \boldsymbol{S}^1 \smallsetminus \{1\}$. Let

$$\Phi\colon U_{-1} \to \mathbf{R} \quad \text{and} \quad \Psi\colon U_1 \to \mathbf{R}$$

be the diffeomorphisms given by

$$\Phi(e(\theta)) = \tan(\theta/2), \quad \text{and} \quad \Psi(e(\theta)) = \cot(\theta/2).$$

There is a picture in the next page to help you understand the co-ordinate changes.

1) Show that Φ and Ψ are well defined, i.e., if $e(\theta_1) = e(\theta_2)$ then $\Phi(e(\theta_1)) = \Phi(e(\theta_2))$ and $\Psi(e(\theta_1)) = \Psi(e(\theta_2))$. Show also that the map

$$\Psi \circ \Phi^{-1}\colon \mathbf{R} \smallsetminus \{0\} \to \mathbf{R} \smallsetminus \{0\}$$

is $y \mapsto 1/y$.

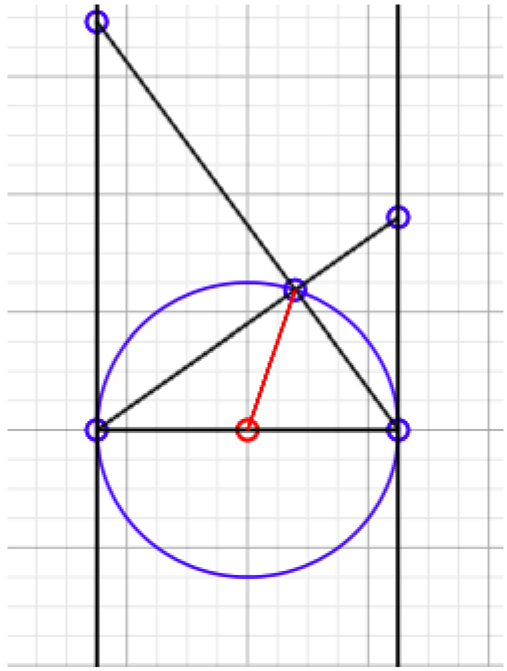
2) Consider the vector field $\boldsymbol{v}(e(\theta)) = (\cos\theta + \sin\theta - 1)\frac{d}{d\theta}$ on $\boldsymbol{S}^1$. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the map $y \mapsto y(1-y)$ and $g\colon \mathbf{R} \to \mathbf{R}$ the map $z \mapsto 1 - z$. Show that

$$\Phi_*(\boldsymbol{v}|_{U_{-1}}) = f\boldsymbol{\vartheta},$$

and

$$\Psi_*(\boldsymbol{v}|_{U_1}) = g\boldsymbol{\vartheta}.$$

Please see the picture on the next page to help you with the above exercises as well as the later exercises. The distance between the two marked points on the line $x = -1$ is $2\Psi(e(\theta))$ and the distance between the marked points on $x = 1$ is $2\Phi(e(\theta))$.



Let $g^t\colon \boldsymbol{S}^1 \to \boldsymbol{S}^1$ be given by

$$g^t(e(\theta)) = e\left\{2\arctan\left(\frac{e^t\tan(\frac{\theta}{2})}{e^t\tan(\frac{\theta}{2}) - \tan(\frac{\theta}{2}) + 1}\right)\right\}$$

with the understanding that $\arctan(\infty) = \pi/2$ and $\arctan(-\infty) = -\pi/2$, or $g^t(e(\theta)) = e(\pi) = e(-\pi) = -1$ when $t = \ln\left(\frac{\tan(\theta/2)-1}{\tan(\theta/2)}\right)$.

3) Show that $g^t(e(\theta))$ is well-defined, i.e., if $e(\theta_1) = e(\theta_2)$ then $g^t(e(\theta_1)) = g^t(e(\theta_2))$.

4) Show that $\{g^t\}$ is a 1-parameter group of diffeomorphisms on $\boldsymbol{S}^1$. [Hint: With $\boldsymbol{v}$ as in Problem **2)**, consider the autonomous first order DE $\dot{\boldsymbol{p}} = \boldsymbol{v}(p)$ on $\boldsymbol{S}^1$. Restrict it to U_{-1} and translate the restricted DE to an autonomous first order DE on $\mathbf{R}$ using Φ. Work out the phase flows for the new DE. Translate back to U_{-1}. See that the formula you get extends to $\boldsymbol{S}^1$.]

5) What are the orbits of $\{g^t\}$ and what are the fixed points for $\{g^t\}$?