HW 7

Due date: March 9, 2021

This HW assignment is largely computational but with very clear theoretical goals. The cookbook problems are to help you understand that as in integral calculus, the issue is one of finding the right set of co-ordinates under which the DE simplifies (the "method of substitution"). In the end, that is the aim of the *rectification theorem* that we will prove later in the course. The remaining problems are to help you with other theoretical topics which will come up.

Cookbook problems.

1) Cookbook-II: 19, 20.

First integrals.

2) Let $\boldsymbol{v}$ be the vector field on $\Omega = \{(x, y, z) \mid y \neq 0\}$ given by

$$\boldsymbol{v} = ((y^2 + z^2)y^{-1},\ xz,\ -xy).$$

Find two first integrals f and g for $\boldsymbol{v}$ on a suitable large open subset U of Ω such that their level surfaces intersect transversally (i.e., the Jacobian matrix $\partial(f, g)/\partial(x, y, z)$ is of rank 2 at every point of U). Specify U.

3) Consider the vector field $\boldsymbol{v}$ on $\mathbf{R}^3$ given by

$$\boldsymbol{v} = \begin{bmatrix} y+z \\ \\ y \\ \\ x-y \end{bmatrix}$$

Let U be the open subset of $\mathbf{R}^3$ on which $z^2 > (x - y)^2$ and $y > 0$.

(a) Find two first integrals f and g for $\boldsymbol{v}$ on U such that their level surfaces intersect transversally (i.e. the Jacobian matrix $\partial(f, g)/\partial(x, y, z)$ is of rank 2 at every point of U). [**Hint:** Show that $\frac{\mathrm{d}(x+z)}{\mathrm{d}y} = \frac{x+z}{y}$ and $\frac{\mathrm{d}(x-y)}{\mathrm{d}z} = \frac{z}{x-y}$.]

(b) If (x_0, y_0, z_0) a point in U, use the first integrals you found for $\boldsymbol{v}$ to find the solution to the IVP

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}) \qquad \boldsymbol{x}(0) = (x_0, y_0, z_0).$$

[**Hint:** If S_1 and S_2 are the two level surfaces $f = f(x_0, y_0, z_0)$ and $g = g(x_0, y_0, z_0)$ respectively, and $C = S_1 \cap S_2$, write C in parametric form using y as the parameter. Now use the equation $\dot{y} = y$ to write y in terms of y_0 and t and substitute to get C in a parametric form with t as a parameter.]

(c) Show that your solution in (b) depends smoothly on (x_0, y_0, z_0).

Equation of variations. Thr Equation of Variations associated with a DE an important theoretical tool in the study of DEs, and we will be studying it soon. This problem gives you a computational understanding of this theoretical tool. Recall that in Example 1.1.1 of Lecture 14 we considered the DE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ xy(z^2+1) \end{bmatrix}$$

and we worked out the solution $\boldsymbol{\varphi}_{\boldsymbol{p}_0}$ for the initial state $\boldsymbol{p}_0 = (x_0,\, y_0,\, z_0)$ in the open set U of $\mathbf{R}^3$ in which the first two coordinates are positive. Specifically, we found that for t in the maximal interval of existence $J_{\max}(\boldsymbol{p}_0)$ for the solution which is $\boldsymbol{p}_0$ at 0, the we have

$$\boldsymbol{\varphi}_{\boldsymbol{p_0}}(t) = \left(x_0 e^t, y_0 e^t, \tan\Big(\frac{1}{2}x_0 y_0 e^{2t} - \frac{1}{2}x_0 y_0 + \arctan z_0\Big)\right).$$

In what follows, let $\boldsymbol{p} = (x, y, z) \in U$. Let $A(t,\, \boldsymbol{p}) = (J\boldsymbol{v})(\boldsymbol{\varphi}_{\boldsymbol{p}}(t))$ for $t \in \mathbf{R}$, where $\boldsymbol{v}(x, y, z) = (x, y, xy(z^2+1))$ and $J\boldsymbol{v}$ the Jacobian of $\boldsymbol{v}$. In explicit terms, if $\theta(t,\, \boldsymbol{p}) = \theta(t, x, y, z) = \frac{1}{2}xye^{2t} - \frac{1}{2}xy + \arctan z$, then

$$A(t, x, y, z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ye^t \sec^2(\theta(t,x,y,z)) & xe^t \sec^2(\theta(t,x,y,z)) & 2xye^{2t}\tan(\theta(t,x,y,z)) \end{bmatrix}$$

It should be pointed out that $J_{\max}(\boldsymbol{p}_0) = \{t \in \mathbf{R} \mid -\pi/2 < \theta(t,\, \boldsymbol{p}_0) < \pi/2\}$.

4) For $\boldsymbol{p} = (x,\, y,\, z) \in U$ and $t \in J_{\max}(\boldsymbol{p})$ let

$$\boldsymbol{y}_1(t) = \left(e^t, 0, \Big(\frac{1}{2}ye^{2t} - \frac{1}{2}y\Big)\sec^2(\theta(t,x,y,z))\right)$$

and

$$\boldsymbol{y}_2(t) = \left(0, e^t, \Big(\frac{1}{2}xe^{2t} - \frac{1}{2}x\Big)\sec^2(\theta(t,x,y,z))\right).$$

(a) Verify that $\boldsymbol{y}_1$ is the solution of the homogeneous linear IVP

$$\dot{\boldsymbol{\zeta}} = A(t, x, y, z)\boldsymbol{\zeta}, \quad \boldsymbol{\zeta}(0) = \mathbf{e}_1$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbf{R}^3$.

(b) Show with minimal computations that $\boldsymbol{y}_2$ is the solution of the IVP

$$\dot{\boldsymbol{\zeta}} = A(t, x, y, z)\boldsymbol{\zeta}, \quad \boldsymbol{\zeta}(0) = \mathbf{e}_2.$$

[**Hint:** Theorem 2.1.3 of Lecture 11 of ANA2 may help.]

(c) What is the solution of the IVP: $\dot{\boldsymbol{\zeta}} = A(t, x, y, z)\boldsymbol{\zeta}$, $\boldsymbol{\zeta}(0) = \mathbf{e}_3$? You are expected to simply write down your answer without justification. If you have done part (b) correctly, you will have no difficulty with this.