HW 6

Due date: None

Do these problems. They may help you understand many things before the exam. You don't have to submit these problems. But you can ask for help with them from your TAs.

Cookbook problems.

1) Cookbook-II: 8, 18.

Lipschitz Conditions.

2) Let Ω be an open domain in $\mathbf{R} \times \mathbf{R}^n$ and let $\boldsymbol{v}\colon \Omega \to \mathbf{R}^n$ be a $\mathscr{C}^1$ function. Show that $\boldsymbol{v}$ is locally Lipschitz in the second argument.

One-parameter groups. By a one-parameter group of diffeomorphisms we mean the one in (1.3.3) of Lecture 9 (i.e. the "tweak" and not the one in (1.2.1) of *loc.cit.*).

In what follows $\boldsymbol{v}\colon U \to \mathbf{R}^n$ is a locally Lipschitz map on an open subset U of $\mathbf{R}^n$, and $J_{\boldsymbol{a}}$ is the maximal interval of existence of the IVP $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{a}$. The maximal solution of the IVP is denoted $\boldsymbol{\varphi_a}$. Note that $\boldsymbol{\varphi_a}$ is a map from $J_{\boldsymbol{a}}$ to U.

To do the problems, you may assume the following theorem which will be proved one day in class.

Theorem 1. *Let W be an open subset of $\mathbf{R} \times U$ contained in the set*

$$\{(t,\, \boldsymbol{a}) \in \mathbf{R} \times U \mid t \in J_{\boldsymbol{a}}\}.$$

Suppose $\boldsymbol{v}$ is $\mathscr{C}^1(U,\, \mathbf{R}^n)$. Then the map $(t,\, \boldsymbol{a}) \mapsto \boldsymbol{\varphi_a}(t)$ is a $\mathscr{C}^1$ map on W.

3) Suppose $J_{\boldsymbol{a}} = \mathbf{R}$ for all $\boldsymbol{a} \in U$. Show that $\{g^t\}$ is a one parameter group of transformations on U, where $g^t(\boldsymbol{x}) = \boldsymbol{\varphi_x}(t)$ for $\boldsymbol{x} \in U$ and $t \in \mathbf{R}$.

4) Suppose $\boldsymbol{v}$ is $\mathscr{C}^1$, and $J_{\boldsymbol{a}} = \mathbf{R}$ for every $\boldsymbol{a} \in U$. Show that the one-parameter group in **3**) is a one-parameter group of diffeomorphisms.

5) Show that there is a one parameter group $\{g^t\}$ of diffeomorphisms on the interval $(-\frac{\pi}{2},\, \frac{\pi}{2})$ such that its phase velocity is given by $v(x) = \cos x$.

6) Argue using the theory of differential equations that

$$\lim_{x \to -\frac{\pi}{2}^+} (\sec x + \tan x) = 0.$$

7) Consider the differential equation $\dot{x} = \operatorname{sgn}(x)\sqrt{|x|}$. Note that $x = \frac{1}{4}t^2$ and $x \equiv 0$ are both solutions which satisfy the initial condition $x(0) = 0$. Explain why this does not violate any theorems you know.