HW 5

Due date: February 12, 2021

Eigenvalues and Jordan forms. Let A be the $n \times n$ real matrix:

$$A = \begin{bmatrix} 0 & 1 & & \ddots & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & \ddots & 1 \\ a_0 & & & \dots & a_{n-1} \end{bmatrix}$$

1) Show that the characteristic polynomial of A is, up to sign,

$$T^n - a_{n-1}T^{n-1} - \cdots - a_1 T - a_0$$

2) Suppose λ is an eigenvalue of A. Show that the eigenspace V_λ of λ is one dimensional and spanned by

$$\begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{bmatrix}.$$

Conclude that there is exactly one Jordan block for each distinct eigenvalue of A. (You may use the fact that the number of Jordan blocks associated with an eigenvalue is equal to the geometric multiplicity of the eigenvalue)

Exponentials.

3) Find e^A for the real matrices A given below.

(a) $A = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$.

(b) $A = \left[\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right]$, where $\theta \in \mathbf{R}$.

4) Let I_2 be the identity 2×2 matrix, $a, b \in \mathbf{R}$ and M the matrix

$$M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

over $\mathbf{R}$ and let $\lambda \in \mathbf{R}$. Let A be the $2n \times 2n$ real matrix:

$$A = \begin{bmatrix} M & I_2 & & & 0 \\ & M & I_2 & & \\ & & M & \ddots & \\ & & & \ddots & I_2 \\ & & & & M \end{bmatrix}$$

Show that for some 2×2 matrix B

$$e^{tA} = \begin{bmatrix} B & tB & \frac{t^2}{2!}B & \dots & \frac{t^{n-1}}{(n-1)!}B \\ & B & tB & \dots & \frac{t^{n-2}}{(n-2)!}B \\ & & B & \dots & \frac{t^{n-3}}{(n-3)!}B \\ & & & \ddots & \frac{t^2}{2!}B \\ & & & & tB \\ & & & & B \end{bmatrix}.$$

Compute B (giving all four entries of B).

One parameter groups. Let M be the open interval $(0, 1)$ in $\mathbf{R}$.

5) Find a one-parameter group of diffeomorphisms $\{g^t\}$ on M such that its associated autonomous differential equation is

$$\dot{x} = x(1 - x).$$

6) For $t \in \mathbf{R}$ and $x \in M$ define

$$g^t x = \frac{x}{x + (1 - x)e^t},$$

Show that $g^t x \in M$ and that $\{g^t\}$ is a one-parameter group of diffeomorphisms on M. Find the phase velocity field of $\{g^t\}$ and write down the associated autonomous differential equation.

7) Sketch the integral curves in the extended phase space of the one-parameter groups occurring in problems (5) and (6) above. (See Definition 1.1.3 of Lecture 9 for the definitions of an integral curve and the extended phase space associated to a one-parameter group.)