HW 4

Due date: February 8, 2021

Variation of Parameters. Use variation of parameters to find a particular solution of the following equations (after converting them into first order vector DEs).

1) $x^3\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2xy = 2,\ x > 0$

2) $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\dfrac{\mathrm{d}y}{\mathrm{d}t} + y = \dfrac{e^t}{1+t^2}$

3) $(1-x^2)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - \dfrac{1}{x}\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\sqrt{1-x^2}$, $0 < x < 1$. (Note that $y \equiv 1$ is a solution of the associated homogeneous equation. See the section under the heading "Second order equations with the dependent variable missing" in Cookbook-1 to find a basis for the solutions of the associated homogeneous equation.)

Some inequalitites. Let I be an interval, t_0 a point in I, a and b continuous real-valued functions on I. Recall that the solution of the linear first order IVP

$$\dot{u}(t) = a(t)u(t) + b(t), \quad u(t_0) = u_0$$

involved an integrating factor. Specifically, if we write $p(t) = \int_{t_0}^{t} a(s)ds$, then

$$\frac{d}{dt}\Big(e^{-p(t)}u(t)\Big) = e^{-p(t)}b(t),$$

which implies, upon applying the operator $\int_{t_0}^{t}(-)(s)ds$ to both sides

$$e^{-p(t)}u(t) - u_0 = \int_{t_0}^{t} e^{-p(s)}b(s)ds.$$

From here one gets the solution

$$u(t) = u_0e^{p(t)} + \int_{t_0}^{t} e^{p(t)-p(s)}b(s)ds.$$

Here is a twist to this solution.

4) Suppose $I = [t_0, t_1]$ and suppose a and b are continuous real-valued functions on I. Let $p(t) = \int_{t_0}^{t} a(s)ds$. Suppose $u\colon I \to \mathbf{R}$ is $\mathscr{C}^1$ and satisfies

$$\left.\begin{array}{rcl} \dot{u}(t) & \leq & a(t)u(t) + b(t) \\ u(t_0) & = & u_0 \end{array}\right\} \quad (t \in I)$$

Show that

$$u(t) \leq u_0e^{p(t)} + \int_{t_0}^{t} e^{p(t)-p(s)}b(s)ds.$$

5) Suppose $I = [t_0, t_1]$ and φ, ψ, and α are continuous real-valued functions on I, with $\alpha \geq 0$. Suppose

$$\varphi(t) \leq \psi(t) + \int_{t_0}^{t} \alpha(s)\varphi(s)ds \qquad (t \in I).$$

Let $q(t) = \int_{t_0}^{t} \alpha(s)ds$. Show that

$$\varphi(t) \leq \psi(t) + \int_{t_0}^{t} e^{q(t)-q(s)}\alpha(s)\psi(s)ds.$$

[Hint: Use Problem (**4**).] (Note that in particular if ψ is a constant, say $\psi(t) \equiv c$, then the inequality above says $\varphi(t) \leq ce^{q(t)}$.)

6) Suppose Ω is a subset of $\mathbf{R} \times \mathbf{R}$ and $f\colon \Omega \to R$ a continuous function which is Lipschitz continuous in the second variable with Lipschitz constant L.[1] Let $t_0 \in \mathbf{R}$ and let I be an interval of the form $[t_0, b]$, or $[t_0, b)$, or $[t_0, \infty)$. Suppose u and v are C^1 functions on I such that $(t, u(t))$ and $(t, v(t))$ are in Ω for all $t \in I$. Suppose further that u and v satisfy

$$\begin{aligned} \dot{u}(t) &\leq f(t, u(t)), \\ \dot{v}(t) &= f(t, v(t)), \\ u(t_0) &\leq v(t_0). \end{aligned}$$

Show that $u(t) \leq v(t)$ for $t \in I$. [Hint: Assume the contrary. Show that $\dot{u} - \dot{v} \leq u - v$ on a suitable sub-interval of of the form $J = [t_1, \infty) \cap I$ with $u(t_1) = v(t_1)$ and $u \geq v$ on J. Use Problem (**4**) to derive a contradiction.]

7) In the previous problem show that the conclusion holds with a slightly different set of hypotheses. Let f be continuous on Ω. We no longer assume that f is Lipschitz in the second variable on all of Ω, but only on a subset V of Ω which has the property that

$$(t, x) \in V \Longrightarrow (\{t\} \times [x, \infty)) \cap \Omega \subset V.$$

Assume that $(t, v(t)) \in V$ for all $t \in I$ (we do not make this extra assumption on u; however we continue to assume $(t, u(t)) \in \Omega$ for $t \in I$). With these changes, and with the rest of the hypotheses the same, show that the conclusion of the previous problem holds. [Hint: Examine the proof you gave for the previous problem carefully.]

Definition 1. *We say that* $\boldsymbol{\varphi}$ *is an* ϵ-approximate solution *of the DE* $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$ *on an interval* I *if* $(t, \boldsymbol{\varphi}(t))$ *is in the domain of* $\boldsymbol{v}$ *for all* $t \in I$ *and*

$$\left\| \dot{\boldsymbol{\varphi}}(t) - \boldsymbol{v}(t, \boldsymbol{\varphi}(t)) \right\| \leq \epsilon \qquad (t \in I).$$

We will be using the following fundamental estimate quite heavily in the proof that solutions of DE's vary smoothly with the initial conditions.

[1] i.e for fixed t, the inequality $|f(t,x) - f(t,y)| \leq L|x-y|$ holds for all $(t,x), (t,y) \in (\{t\} \times \mathbf{R}) \cap \Omega$. We have also used the term "Lipschitz in phase space" for this.

8) (The Fundamental Estimate) Let $\Omega \subset \mathbf{R} \times \mathbf{R}^n$ and $\boldsymbol{v}\colon \Omega \to \mathbf{R}^n$ a continuous function which is Lipschitz in the second variable with Lipschitz constant L. Suppose $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are $\mathscr{C}^1$ functions on an interval I, with $\boldsymbol{\varphi}$ an ϵ_1-approximation of $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$ on I and $\boldsymbol{\psi}$ an ϵ_2-approximation of $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$ on I. Suppose further that at a specified point t_0 in I we have $\|\boldsymbol{\varphi}(t_0) - \boldsymbol{\psi}(t_0)\| \le \delta$. Then

$$\Big\|\boldsymbol{\varphi}(t) - \boldsymbol{\psi}(t)\Big\| \le \delta e^{L|t-t_0|} + \frac{\epsilon_1 + \epsilon_2}{L}\left(e^{L|t-t_0|} - 1\right) \qquad (t \in I).$$

[Hint: Let $u(t) = \|\boldsymbol{\varphi}(t) - \boldsymbol{\psi}(t)\|^2$. Let $\epsilon = \epsilon_1 + \epsilon_2$. Show that $\dot{u} \le 2Lu + 2\epsilon\sqrt{u}$. Show that $f(x) = 2Lx + 2\epsilon\sqrt{x}$ is Lipschitz on $[\delta^2, \infty)$ by using the fact that $\dfrac{df}{dx} = 2L + \dfrac{\epsilon}{\sqrt{x}}$ is bounded in $[\delta^2, \infty)$. Now use Problem (**7**).]