HW 3

Cookbook problems. Solve:

1) Cookbook-I: **13** and **22**.

2) Cookbook-II: **3** and **9**.

The operator norm on matrices. If V and W are vector spaces over a field F, we write $\mathrm{Hom}_F(V,W)$ for the vector space of linear transformations from V to W. Recall from §§2.1 of Lecture 5 in ANA2 that the vector space of linear operators from $\mathbf{R}^n$ to $\mathbf{R}^n$ has a norm. The norm was denoted there by $\| \, \|_L$. However, in this course, L will usually be used for Lipschitz constants and hence we will use the symbol $\| \, \|_\circ$ for the operator norm. Recall also that we have $\|AB\|_\circ \leq \|A\|_\circ \|B\|_\circ$ for $A \in \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^k, \mathbf{R}^m)$ and $B \in \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^k)$ (see **7** of § 1 of Lecture 7 in ANA2).

In this HW, if we say I is an interval, then we allow I to be open, closed, or half-open, but insist that it have a non-empty interior. This means single points are not intervals for us.

We often write $M_{m,n}(F)$ for the space of $m \times n$ matrices with entries in a field F. Clearly $M_{m,n}(F)$ can be identified with $\mathrm{Hom}_F(F^n, F^m)$, and we will often do so implictly. Thus it makes sense to talk about the operator norm on $M_{m,n}(\mathbf{R})$, and we will denote this also by $\| \, \|_\circ$. Since $M_{m,n}(\mathbf{R})$ (which we identify with $\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^m)$) is finite dimensional, all norms on it are equivalent. In particular, a map from an interval I to $(M_{m,n}(\mathbf{R}), \| \, \|_\circ)$, say $A\colon I \to M_{m,n}(\mathbf{R})$, is continuous if and only if each of the entries of the matrix of functions A is continuous.[1] Let $t_\circ$ be a point in I. We say $A\colon \to M_{m,n}(\mathbf{R})$ is *differentiable at* $t_\circ$ if:

$$(*) \qquad \lim_{h \to 0} \frac{1}{h}\Big(A(t_\circ + h) - A(t_\circ)\Big)$$

exists. The above limit is the appropriate one-sided limit in the event our interval is closed or half open, and $t_\circ$ is a boundary point of I. These limits are of course taken with respect to $\| \, \|_\circ$. However, as we just argued, any norm can be used, since all norms on $M_{m,n}(\mathbf{R})$ are equivalent. If A is differentiable at $t_\circ$ then the limit in $(*)$ is called the *derivative* of A at $t_\circ$ and denoted by any of the familiar symbols $\frac{\mathrm{d}A}{\mathrm{d}t}|_{t=t_\circ}$, $\dot{A}(t_\circ)$, $\frac{\mathrm{d}A}{\mathrm{d}t}(t_\circ)$, $A'(t_\circ)$ etc. A is said to be *differentiable on* I if it is differentiable at every point of I. In such a case, the function $t \mapsto \dot{A}(t)$ is called the derivative of A on I, and is denoted by the symbols $\dot{A}$, $\frac{\mathrm{d}A}{\mathrm{d}t}$, A', and sometimes as $\frac{\mathrm{d}A(t)}{\mathrm{d}t}$ or $\frac{\mathrm{d}}{\mathrm{d}t}A(t)$.

[1] A function $A\colon S \to M_{m,n}(\mathbf{R})$, where S is some non-empty set, is the same as an $m \times n$ matrix of functions $A = (a_{ij})$ with $a_{ij}\colon S \to \mathbf{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$.

3) Let I be an interval in $\mathbf{R}$ and suppose

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) \qquad (t \in I)$$

is a linear first order differential equation with A an $n \times n$ matrix of continuous functions on an I. Suppose $\boldsymbol{y}_1(t), \ldots, \boldsymbol{y}_n(t)$ are n solutions of this differential equation on I. Let

$$W = W(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n) \colon I \longrightarrow \mathbf{R}$$

be the function given by

$$W(t) = \det\left[\boldsymbol{y}(t), \ldots, \boldsymbol{y}_n(t)\right] \qquad (t \in I),$$

where $[\boldsymbol{y}(t), \ldots, \boldsymbol{y}_n(t)]$ is regarded an $n \times n$ matrix whose i^{th} column is $\boldsymbol{y}_i(t)$, $i = 1, \ldots, n$. Show that either W is identically zero on I or it is nowhere vanishing on I.

4) Let I be an interval in $\mathbf{R}$. Suppose $A \colon I \to M_{m,k}(\mathbf{R})$ and $B \colon I \to M_{k,n}(\mathbf{R})$ are differentiable on I. Show that $t \mapsto A(t)B(t)$ gives us a differentiable map $AB \colon I \to M_{m,n}(\mathbf{R})$ and that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(A(t)B(t)\Big) = \dot{A}(t)B(t) + A(t)\dot{B}(t) \qquad (t \in I).$$