HW 2

You may use the following theorem (the proof of which will be done some time in the course):

Theorem 1. *Suppose $p_i(t)$, $i = 1, \dots, n$ and $g(t)$ are continuous real-valued functions on an interval (a, b) that contains the point t_0. Then for any choice of real numbers $\alpha_0, \dots, \alpha_{n-1}$, there exists a unique solution $y(t)$ on the interval (a, b) to the ordinary linear differential equation*

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \cdots + p_n(t)y(t) = g(t)$$

for all $t \in (a, b)$ with $y^{(i)}(t_0) = \alpha_i$, for $i = 0, \dots, n-1$.

Cookbook problems. Solve:

1) Problems **1** and **12** from Cookbook-I

2) Problems **2** and **14** from Cookbook-II.

3) For what values of r is $y = t^r$ a solution of $t^2y'' + 4ty' - 10y = 0$?

4) Show that the space of solutions to the differential equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0$$

in the interval (a, b), where p_i are continuous on (a, b), is an $\mathbf{R}$-vector space of dimension n.

Vector valued linear differential equations. We identify $\mathbf{R}^n$ with column vectors $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Let

$$A(t) = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & \dots & a_{2n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix}$$

be an $n \times n$ real matrix of continuous real-valued functions $a_{ij}(t)$, $1 \leq i, j \leq n$ on an interval (a, b). Regard a column of continuous functions $\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ on (a, b) as a continuous map $\boldsymbol{x} \colon (a, b) \to \mathbf{R}^n$ in the usual way. We say $\boldsymbol{x}$ is k-times differentiable on (a, b) if each $x_i(t)$ is k-times differentiable on (a, b) and in this case $\frac{d^k \boldsymbol{x}(t)}{dt^k} = \boldsymbol{x}^{(k)}(t)$ is defined to be the vector whose entries the k-fold derivatives of $x_i(t)$, $i = 1, \dots, n$. The first derivative of $\boldsymbol{x}$, if it exists, is often denoted $\dot{\boldsymbol{x}}$, or sometimes $\dot{\boldsymbol{x}}(t)$. Consider the equation

$$(*) \qquad \dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t).$$

Equation $(*)$ is called a *a vector valued first order linear differential equation*, or an $\mathbf{R}^n$*-valued first order linear differential equation.*

Later in the course we will prove the following theorem (you may assume it for now).

Theorem 2. *Let t_0 be a point in (a, b). Then for each $\boldsymbol{\alpha} \in \mathbf{R}^n$, there exists a unique solution $\boldsymbol{x}(t)$ in (a, b) to $(*)$ such that $\boldsymbol{x}(t_0) = \boldsymbol{\alpha}$.*

5) Show that the solutions to $(*)$ form an n-dimensional real vector space.

The constant A case. In the above situation suppose $(a, b) = \mathbf{R}$ and A is a constant matrix of functions. For each $s \in \mathbf{R}$, define a map

$$g(s)\colon \mathbf{R}^n \to \mathbf{R}^n$$

in the following way:

$$g(s)(\boldsymbol{\alpha}) = \boldsymbol{x}(s), \qquad (\alpha \in \mathbf{R}^n)$$

where $\boldsymbol{x}\colon \mathbf{R} \to \mathbf{R}^n$ is the unique solution to $(*)$ with $\boldsymbol{x}(0) = \boldsymbol{\alpha}$. Show:

6) The map $g(s + t) = g(s)g(t)$ for all s and t in $\mathbf{R}$, and $g(0)$ is the identity map.

7) Each $g(s)$ is an invertible linear transformation. (This, combined with the previous problem, shows that g is a group homomorphism from $\mathbf{R}$ to the group of invertible linear transformations of $\mathbf{R}^n$.)

8) Identifying linear endomorphisms of $\mathbf{R}^n$ with $n \times n$ matrices, show that

$$\lim_{t \to 0} \frac{g(t) - g(0)}{t} = A.$$

9) Show that every n-th order linear differential equation of the form

$$(**) \qquad y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \cdots + p_n(t)y(t) = 0$$

where p_i are continuous functions on an interval (a, b), is equivalent to a first order linear $\mathbf{R}^n$-valued linear differential equation of the form $(*)$ on (a, b) such that the if $\boldsymbol{x}(t)$ is a solution to $(*)$ on (a, b), then $x_1(t)$ is a solution to $(**)$, where x_1 is the first co-ordinate of $\boldsymbol{x}$. Conclude that Theorem 1 is a special case of Theorem 2.