COMPACT RIEMANN SURFACES
HW

If a problem is labelled B2 then it is meant only for B.Sc 2nd years.

If $\gamma\colon [\alpha, \beta]$ is a path in $\mathbb{C}$ and ϕ is a non-negative real-valued measurable function on the image of γ, then we write $\int_\gamma \phi(z)|dz|$ as a shorthand for $\int_\alpha^\beta \phi(z)|\gamma'(t)|\,dt$. In many of the following exercises you may need the following version of the Dominated Convergence Theorem (or DCT for short) for complex integrals, and you are allowed to use this theorem:

The Dominated Convergence Theorem (DCT) *Let γ be a path in $\mathbb{C}$ and $\{\phi_n\}$ a sequence of measurable complex-valued functions on the image of γ such that (a) $\phi_n \to \phi$ pointwise as $n \to \infty$, (b) $\int_\gamma |\phi_n(z)||dz| < \infty$, and (c) there exists a measurable function Ψ on the image of γ such that $|\phi_n| \leq |\Psi|$ for all n and $\int_\gamma |\Psi_n(z)||dz| < \infty$. Then*

$$\lim_{n\to\infty} \int_\gamma \phi_n(z)dz = \int_\gamma \phi(z)dz$$

You may assume all functions that occur in these exercises are measurable, because they will be continuous on the image of γ.

Basic function theory

In what follows you may assume results from the notes posted (notes1.pdf) on basic function theory, except the generalized Cauchy's theorem, and the result in Remark 2.5.5 of those notes, or a result specifically eschewed, or a result from the notes that you are specifically asked to prove.

(1) (B2) If $f(z)$ is analytic on a domain Ω and $u(z)$ and $v(z)$ are the real and imaginary parts of $f(z)$ show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

on Ω.

(2) Suppose C is a circle of radius $r > 0$ centered at $a \in \mathbb{C}$, and $\varphi\colon C \to \mathbb{C}$ a holomorphic function. Let D be the open disc bounded by C. Show that $f\colon D \to \mathbb{C}$ defined by

$$f(z) := \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta)d\zeta}{\zeta - z}$$

is holomorphic on D. [Hint: You need a lower bound for the absolute value of the denominator in a neighborhood of z. To differentiate under the integral sign you will need the DCT when you apply the definition of a derivative.]

(3) (B2) Suppose a is the centre of an open disc D such that $f(z)$ is an analytic function in $D \setminus \{a\}$ and $\lim_{z\to a}(z-a)f(z) = 0$ (in other words $f(z)$ has a removable singularity at a). Show that there is a an analytic function $g(z)$ on D which extends $f(z)$. [Hint: Use the result from the previous problem.]

Implicit and Inverse function theorems

Let $f(z)$ be analytic in a neighborhood of z_0 and suppose $w_0 = f(z_0)$ and $f'(z_0) \neq 0$. Pick a disc D centered at z_0 such that $f(z)$ is analytic on its closure $\overline{D}$ and such that z_0 is the only solution of $f(z) = w_0$ in $\overline{D}$. Let C be the circle bounding D with positive orientation. Let γ be the image of C and $U \subset \mathbb{C}$ be the connected component of $\mathbb{C} \setminus \gamma$ containing w_0. For $w \in U$ define $g(w)$ by the integral formula:

$$g(w) = \frac{1}{2\pi i}\int_C \frac{zf'(z)dz}{f(z)-w}.$$

(4) Show that $g(w)$ takes values in D.

(5) Is $f(D) = U$? Why or why not?

(6) Show that $f|_{g(U)} \colon g(U) \to U$ is bijective and its inverse is $g\colon U \to g(U)$. Conclude that f is biholomorphic on $g(U)$.

In the exercises that follow, the statements, as well as the proofs work more or less verbatim for the more general case of an analytic function of two variables defined on an open set of $\mathbb{C}^2$, but I did not want to get into the definitions and properties of analytic functions of two variables, and so am restricting myself to polynomials.

Let $\mathbb{C}[U, V]$ be the polynomial ring in two variables over $\mathbb{C}$. Set

$$\partial_1 := \frac{\partial}{\partial U} \colon \mathbb{C}[U, V] \to \mathbb{C}[U, V]$$

and

$$\partial_2 := \frac{\partial}{\partial V} \colon \mathbb{C}[U, V] \to \mathbb{C}[U, V].$$

Let $p \in \mathbb{C}[U, V]$ be a polynomial in two variables and suppose $(z_0, w_0) \in \mathbb{C}^2$ is a point such that $p(z_0, w_0) = 0$ and $\partial_2 p(z_0, w_0) \neq 0$. Fix an open disc D centered at w_0 such that w_0 is the only solution of $p(z_0, w) = 0$ on the closure $\overline{D}$ of D (this is possible since $p(z_0, w)$ is a polynomial of w since z_0 is fixed). Let C be the circle bounding D and assume C is given its positive orientation (i.e., counter-clockwise). Let

$$G = \{z \in \mathbb{C} \mid p(z, w) \neq 0, \ \forall w \in C\}.$$

By Exercise (7) below, G is open. Let G_0 be the *connected component* of G containing z_0.

(7) Let K be a non-empty compact set in $\mathbb{C}$. Show that the set

$$S := \{z \in \mathbb{C} \,|\, p(z,\, w) \neq 0,\ \forall w \in K\}$$

is open in $\mathbb{C}$. [Hint: This only needs the continuity of p.]

(8) (a) For $z \in G$ show that

$$n(z) := \frac{1}{2\pi i} \int_C \frac{\partial_2 p(z,w)dw}{p(z,w)}$$

is constant on connected components of G. [Hint: Use DCT to show that $z \mapsto n(z)$ is continuous. How does that help?]

(b) What is the value of $n(z)$ on G_0?

(9) Fix z^*. Show that there is a closed disc Δ centered at $0 \in \mathbb{C}$ such that

$$(h,w) \mapsto \frac{w}{h} \left[\frac{\partial_2 p(z^*+h,w)p(z^*,w) - \partial_2 p(z^*,w)p(z^*+h,w)}{p(z^*+h,w)p(z^*,w)} \right]$$

is bounded on $\Delta \times C$.

(10) Show that the function $f\colon G_0 \to \mathbb{C}$ defined by

$$f(z) := \frac{1}{2\pi i} \int_C \frac{w\partial_2 p(z,w)dw}{p(z,w)}$$

is holomorphic, takes values in D, and

$$p(z,\, f(z)) \equiv 0$$

on G_0. [Note: This is the implicit function theorem for holomorphic functions.]

In the next few question f, C, D, γ, g, U etc be as in Problems 4, 5, and 6. Let $\Gamma \subset \overline{D}$ be $\Gamma := f^{-1}(\gamma) \cap \overline{D}$, where we regard γ as a subset of $\mathbb{C}$ rather than as a path. Note that $C \subset \Gamma$, if we regard C as a set rather than as a path. Let $R := \overline{D} \setminus \Gamma$. Note that R is an open subset of D and hence of $\mathbb{C}$. Let $V \subset R$ be the connected component containing z_0.

(11) Show that

$$z \mapsto \frac{1}{2\pi i} \int_C \frac{f'(\zeta)d\zeta}{f(\zeta) - f(z)}$$

is constant on connected components of $\overline{D} \setminus \Gamma$. What is its value on V?

(12) Show that

$$z \mapsto \frac{1}{2\pi i} \int_C \frac{\zeta f'(\zeta)d\zeta}{f(\zeta) - f(z)}$$

is holomorphic on R. What is it on V?

(13) Show $g(U) = V$, and that $f|_U$ and g are inverses of each other.

The problems below are related to Problems 7, 8, 9, and 10. Let

$$p(U, V) = a_0(U)V^n + a_1(U)V^{n-1} + \cdots + a_{n-1}(U)V + a_n(U)$$

where $a_i(U) \in \mathbb{C}[U]$ for $i = 0, \ldots, n$. Let $z_0 \in \mathbb{C}$ be a complex number such that $p(z_0, V) \in \mathbb{C}[V]$ has n *distinct* roots, $w_1^*, \ldots, w_n^*$ in $\mathbb{C}$ (note that this means $a_0(z_0) \neq 0$.

(14) Show that there is an open neighborhood $\mathfrak{U}$ of z_0 and n holomorphic functions $w_1, \ldots, w_n$ on $\mathfrak{U}$ such that for $i, j \in \{1, \ldots, n\}$ we have
- $w_i(z_0) = w_i^*$,
- $p(z, w_i(z)) \equiv 0$ on $\mathfrak{U}$, and
- $w_i(\mathfrak{U}) \cap w_j(\mathfrak{U}) = \emptyset$ if $i \neq j$.

(15) Let $\mathfrak{U}$ and $w_i\colon \mathfrak{U} \to C$, $i = 1, \ldots, n$ be as in Problem 14 and by shrinking $\mathfrak{U}$ if necessary, assume $\mathfrak{U}$ is connected. Show that if $w\colon \mathfrak{U} \to \mathbb{C}$ is any holomorphic function such that $p(z, w(z)) \equiv 0$ on $\mathfrak{U}$, then we have a functional identity $w = w_i$ for some $i \in \{1, \ldots, n\}$.

Basic Riemann Surfaces

Suppose X is a Riemann surface and $z\colon U \to V$ is a coordinate chart on X (in other words V is an open subset of $\mathbb{C}$ and U and V are isomorphic via z). Suppose $f\colon U \to \mathbb{P}^1$ is a meromorphic function. Let $g\colon V \to \mathbb{P}^1$ be the meromorphic function $f \circ z^{-1}$. Then f is a holomorphic function on $U \setminus \Sigma$, where Σ is a discrete subset of U. Note that $f(\Sigma)$ is discrete in V and g is holomorphic function on $V \setminus f(\Sigma)$. Let g' be the derivative of g on $V \setminus f(\Sigma)$. Then $(g', f(\Sigma)$ represents a meromorphic function on V which we again denote g'.

Define the *derivative of f with respect to z* on U to be the meromorphic function given by the formula:

$$\frac{df}{dz} := g' \circ z.$$

Note that $\frac{df(z)}{dz}$ is meromorphic on U and holomorphic on $U \setminus \Sigma$. For the next three problems this notation will be regarded as fixed.

(16) Let $z^*\colon U \to W$ be another chart on the same open set U.

(a) Show that

$$\frac{df}{dz^*} = \frac{df}{dz}\frac{dz}{dz^*}.$$

(b) Show that $\frac{dz}{dz^*}$ is nowhere vanishing on U and

$$\frac{1}{\frac{dz}{dz^*}} = \frac{dz^*}{dz}$$

on U.

(17) Suppose $x \in U$ and $a = z(x)$.

(a) Show that f can be expanded as a Laurent series in $z - a$ in an open neighbourhood of a in U in such a way that the principal part of the Laurent series is a finite series. [Recall: If $\sum_{-\infty}^{\infty} p_n(z-a)^n$ is a Laurent series, then its principal part is $\sum_{-\infty}^{-1} p_n(z-a)^n$.]

(b) Without loss of generality, assume that the open neighbourhood on which the Laurent series expansion asserted above is the open set U. Say $f = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$. Show that the derivative of f with respect to z is given by $\sum_{n=-\infty}^{\infty}(n+1)a_{n+1}(z-a)^n$. You may assume (for both parts) analogous theorems for holomorphic functions on the complex plane.

(18) Suppose $x \in U$ and $a = z(x)$. Let $f = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$ be the Laurent expansion of f with respect to $z - a$. Define the *order of f at x*, denoted $\mathrm{ord}_x(f)$, to be the integer n such that $a_i = 0$ for all $i < n$ and $a_n \neq 0$.
(a) Show that $\mathrm{ord}_x(f)$ is independent of the local chart $z\colon U \to V$.
(b) Show that $\{x \mid \mathrm{ord}_x(f) \neq 0\}$ is a discrete set.

Let $\mathbf{P}$ denote the space $(\mathbb{C}^2 \setminus \{0\})/\sim$, where $\sim$ is the equivalence relation $(x, y) \sim \lambda(x, y)$ for $(x, y) \in \mathbb{C}^2 \setminus \{0\}$ and $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Let $[x, y]$ denote the equivalence class of $(x, y) \in \mathbb{C}^2 \setminus \{0\}$. Set $0_{\mathbf{P}} = [0, 1]$ and $\infty_{\mathbf{P}} = [1, 0]$. Let us also agree to denote by $\mathbb{S}$ the Riemann-sphere, with its standard complex structure. Let S and N denote the South and North poles respectively. Let

$$\pi_N\colon \mathbb{S} \setminus N \to \mathbb{C}$$

denote the stereographic projection from N and

$$\pi_S\colon \mathbb{S} \setminus S \to \mathbb{C}$$

the stereographic projection from S followed by complex conjugation. The complex plane is to be regarded as the *equatorial plane* for both the stereographic projection.

In the same fashion, for $i = 1, 2$ define

$$p_1\colon \mathbf{P} \setminus \infty_{\mathbf{P}} \to \mathbb{C}$$

by $[x, y] \mapsto x/y$ and

$$p_2\colon \mathbf{P} \setminus 0_{\mathbf{P}} \to \mathbb{C}$$

by $[x, y] \mapsto y/x$.

(19) Show that π_S and π_N give a complex structure on $\mathbb{S}$ which makes $\mathbb{S}$ into a compact Riemann surface.

(20) Show that p_i are homeomorphisms where $\mathbf{P}$ is given the standard quotient topology. Show also that $\varphi_1 := p_1^{-1} \circ \pi_N$ and $\varphi_2 := p_2^{-1} \circ \pi_S$ agree on the intersection of their domains, namely on $\mathbb{S} \setminus \{S, N\}$. Conclude that we have a homeomorphism $\varphi\colon \mathbb{S} \to \mathbf{P}$ such that φ restricts to $\varphi_1 (= \varphi_2)$ on $\mathbb{S} \setminus \{S, N\}$.

Note that φ gives a complex structure on $\mathbf{P}$ by transferring the complex structure from $\mathbb{S}$ on to $\mathbf{P}$. This is (obviously) the unique complex structure on $\mathbf{P}$ such that φ is holomorphic. The common Riemann surface structure on $\mathbf{P}$ and $\mathbb{S}$ is what we are calling $\mathbb{P}^1$. For those who know some algebraic geometry, $\mathbf{P}$ is the *projective line*. We are using the symbol $\mathbb{P}^1$ for the Riemann sphere to remind ourselves of its "other role" as the projective line.

(21) Suppose $f\colon X \to Y$ is a non-constant holomorphic map between compact Riemann surfaces of degree n. Show that for any $y \in Y$

$$\sum_{P\in f^{-1}(y)} e_P = n.$$

Differential Forms

For the following problems, we assume familiarity with differential forms on manifolds. If X is a Riemann surface, and ω a 1-form, then ω is said to be a *holomorphic* 1*-form*, or a *holomorphic differential* if in local co-ordinates $\omega = f dz$ where $z\colon U \to V$ is a co-ordinate chart and $f\colon U \to \mathbb{C}$ is holomorphic. In somewhat greater detail, ω is completely determined by data of the form $\{(f_\alpha, z_\alpha)\}_\alpha$, where $z_\alpha\colon U_\alpha \to \mathbb{C}$ are co-ordinate charts on X which give an atlas on X and the relation $f_\beta = f_\alpha \frac{dz_\alpha}{dz_\beta}$ holds on $U_\alpha \cap U_\beta$. There is an obvious equivalence of such data. The pair (f_α, z_α) is to be then regarded as representing $f_\alpha dz_\alpha$.

If the f_α above are meromorphic rather than holomorphic, then ω is said to be a *meromorphic* 1*-form* or a *meromorphic differential*. A holomorphic differential is certainly a meromorphic differential. If ω is a meromorphic differential and P a point of X, then we can talk about the *order of* ω *at* P in the following way: Let $z\colon U \to \mathbb{C}$ be a co-ordinate chart around P and suppose ω is given over U by $f dz$. Then the order of ω at P, denoted $\mathrm{ord}_P(\omega)$, is given by

$$\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f).$$

Suppose ω is a meromorphic differential on X and $P \in X$ a point. Let $z\colon U \to \mathbb{C}$ is a co-ordinate chart on X, with $P \in X$ and $z(P) = 0$. Let ω have the local representation $f dz$ on U, and suppose $\sum_{i=N}^{\infty} a_n z^n$ is the Laurent expansion of f around P (with N an integer, perhaps negative). Define the *residue of* ω *at* P to be

$$\mathrm{res}_P\omega := a_{-1}.$$

A priori the residue depends on the local co-ordinate z. Problem (25) asks you to show that it is independent of the local co-ordinate z.

(22) Show that if ω is a holomorphic form on a Riemann surface X, then $d\omega = 0$.

(23) Show that if X is a compact Riemann surface and $f \in \mathscr{M}(X)$, then

$$\sum_{P\in X} \mathrm{ord}_P(f) = 0.$$

(24) Let ω, ω_1 and ω_2 be meromorphic differentials on a compact Riemann surface X.

(a) Let $P \in X$. Show that $\mathrm{ord}_P(\omega)$ is well defined (i.e., does not depend on the local co-ordinate chosen around P) and is zero for all but a finite number of P. [Note: One does not need compactness to show that the order of ω at P is well-defined.]

(b) Show that

$$\sum_P \mathrm{ord}_P(\omega_1) = \sum_P \mathrm{ord}_P(\omega_2).$$

(25) Let X be a Riemann surface, ω a meromorphic differential on X.

(a) Let $P \in X$. Show that $\mathrm{res}_P(\omega)$ does not depend on the local co-ordinate z chosen around P.

(b) Show that the set $S_\omega = \{x \mid \mathrm{res}_x\omega \neq 0\}$ is discrete

(c) Let X be compact. Show that

$$\sum_P \mathrm{res}_P\omega = 0.$$

Euler characteristic

If X is a compact Riemann surface, it can be trianguated (*Rado's Theorem*). One can show (and we will give an indication in class) that if $\mathbf{T}$ is a triangulation then the number

$$\chi(X) := V_{\mathbf{T}} - E_{\mathbf{T}} + F_{\mathbf{T}}$$

does not depend on $\mathbf{T}$. Here, $V_{\mathbf{T}}$ is the number of vertices in $\mathbf{T}$, $E_{\mathbf{T}}$ is the number of edges in $\mathbf{T}$, and $F_{\mathbf{T}}$ is the number of faces in $\mathbf{T}$. The number $\chi(X)$ is called the Euler characteristic of X.

From Exercise (24) we know that if ω is a meromorphic differential on X, then the integer

$$\Phi(X) := \sum_P \mathrm{ord}_P(\omega)$$

is independent of ω. The aim of the next set of exercises is to show the close connection between $\chi(X)$ and $\Phi(X)$.

(26) Suppose $f\colon X \to Y$ is a non-constant holomorphic map between compact Riemann surfaces of degree n. Show that $e_P - 1$ is zero for all but a finite number of points, and further show that

$$\chi(X) = n\chi(Y) - \sum_{P\in X} (e_P - 1).$$

(27) Suppose $f\colon X \to Y$ is a non-constant holomorphic map between compact Riemann surfaces of degree n. Show that

$$\Phi(X) = n\Phi(Y) + \sum_{P\in X}(e_P - 1).$$

(28) Show using the fact that a compact Riemann surface has enough functions, that

$$\chi(X) = -\Phi(X).$$

[Hint: Compute $\chi(\mathbb{P}^1)$ and $\Phi(\mathbb{P}^1)$.]

Function fields and compact Riemann surfaces

For a function field F over $\mathbb{C}$ we use the symbol $\mathscr{R}(F)$ for the corresponding set of valuation rings of F. And for a compact Riemann surface X we write $\mathscr{M}(X)$ for its field of meromorphic functions. We will accept the theorem guaranteeing enough functions on such an X for the purposes of the HW exercises that follow in this section. The category of compact Riemann surfaces and *non-constant* holomorphic functions will be denoted $\mathscr{X}$ and the category of function fields over $\mathbb{C}$ will be denoted $\mathscr{F}$. For a valuation ring A of a function field $F \in \mathscr{F}$, we write $\mathfrak{m}_A$ for its maximal ideal and identify $A/\mathfrak{m}_A$ with $\mathbb{C}$ via the isomorphism defined by the composite $\mathbb{C} \to A \to A/\mathfrak{m}_A$. The valuation ring A will be identified, as in class, with the map $\varphi_A\colon F \to \mathbb{P}^F$. For $X \in \mathscr{X}$ and $x \in X$, we write $A_x \in \mathscr{R}\mathscr{M}(X)$ for the valuation ring of $\mathscr{M}(X)$ consisting of elements which do not have a pole x. For $f \in F$, $f_*\colon \mathscr{R}(F) \to \mathbb{P}^1$ is the map $A \mapsto \varphi_A(f)$.

The above are mainly reminders of the conventions we established in class, not an exhaustive list of them.

(29) Let $F \in \mathscr{F}$, and $f \in F$. Show that $f_*\colon \mathscr{R}(F) \to \mathbb{P}^1$ is continuous. (We proved it is holomorphic in class, assuming continuity. This exercise is for completeness.)

(30) Let $F \in \mathscr{F}$. Complete the last part of the proof that $\mathscr{R}(F)$ is connected, i.e., show that if $f \in F \setminus \mathbb{C}$ and the fibre cardinality of the holomorphic map $f_*\colon \mathscr{R}(F) \to \mathbb{P}^1$ is almost everywhere m, then $\dim_{\mathbb{C}(f)} \mathscr{M}\mathscr{R}(F) = m$. [Note: As we noted in class, this means: (a) $\mathscr{M}(\mathscr{R}(F))$ is a field, (b) The map $F \to \mathscr{M}\mathscr{R}(F)$ given by $g \mapsto g_*$ is an isomorphism of fields, and (c) $\mathscr{R}(F)$ is connected.]

(31) Let $X \in \mathscr{X}$ and define $p(X)\colon X \to \mathscr{R}\mathscr{M}(X)$ by $x \mapsto A_x$. Show that $p(X)$ is holomorphic.

(32) Let $\alpha\colon F \to G$ be a morphism in $\mathscr{F}$. Show that $\mathscr{R}(\alpha)$ is holomorphic (you will have to first show it is continuous).

The $\overline{\partial}$-derivative

Let X be a Riemann surface and $z\colon U \to \mathbb{C}$ a holomorphic co-ordinate chart on X. Let $z = x + iy$ be the standard decomposition of the co-ordinate z into its real and imaginary parts. Thus $(x, y)\colon U \to \mathbb{R}^2$ will be a real co-ordinate chart on the open set U if X is regarded as a 2-manifold over $\mathbb{R}$. We have the standard vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ on U, as well as differential forms dx and dy. Write $T_X \to X$ and $T_X^* \to X$ for the tangent and cotangent bundles of X. Recall that a (smooth) vector field $\mathbf{v}$ on U is nothing but a C^∞-section of $T_X \to X$ over U, i.e., a C^∞-map $\mathbf{v}\colon U \to T_X$ such that the composite

$$U \xrightarrow{\mathbf{v}} T_X \to X$$

is the identity map on U. In particular $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are such sections of T_X over U. Similarly, (smooth) differential 1-forms over U are C^∞-sections of the co-tangent bundle over U, i.e. C^∞-maps $\omega\colon U \to T_X^*$ such that the composite

$$U \xrightarrow{\omega} T_X^* \to X$$

is the identity on U. Let the space of vector fields over X be denoted $\mathscr{T}(X)$ and the space of differential 1-forms over U by $\mathscr{T}^*(X)$ as well as by$\mathscr{A}^1(X)$. The space of differential 2-forms over X (always C^∞) will be denoted $\mathscr{A}^2(X)$.

Consider the $\mathbb{C}$-vector spaces $\mathscr{T}_{\mathbb{C}}(X) := \mathscr{T}(X) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathscr{A}^1_{\mathbb{C}}(X) := \mathscr{A}^1(X) \otimes_{\mathbb{R}} \mathbb{C}$. These can be regarded as the space of C^∞-sections over X of the complex bundles $T_X \otimes_{\mathbb{R}} \mathbb{C} \to X$ and $T_X^* \otimes_{\mathbb{R}} \mathbb{C} \to X$ (fibre-wise tensor products give us these spaces).

Elements of $\mathscr{T}_{\mathbb{C}}(X)$ are called *complex vector fields* over U and we have a natural decomposition

$$\mathscr{T}_{\mathbb{C}}(X) = \mathscr{T}(X) \oplus i\mathscr{T}(X) \tag{*}$$

of real vector spaces. Similarly, elements of $\mathscr{A}^1_{\mathbb{C}}(X)$ are called *complex* 1*-forms* over X and we have a natural decomposition

$$\mathscr{A}^1_{\mathbb{C}}(X) = \mathscr{A}^1(U) \oplus i\mathscr{A}^1(X) \tag{**}$$

of $\mathbb{R}$-vector spaces.

Let us return to our coordinate chart (U, z). The 1-forms $dx \in \mathscr{A}^1(U)$ and $dy \in \mathscr{A}^1(U)$ allow us to define

$$dz = dx + idy \in \mathscr{A}^1_{\mathbb{C}}(U)$$

and

$$d\bar{z} = dx - idy \in \mathscr{A}^1_{\mathbb{C}}(U)$$

and every complex 1-form $\omega \in \mathscr{A}^1_{\mathbb{C}}(U)$ can be written uniquely as

$$\omega = f(z)dz + g(z)d\bar{z}$$

with f and g *complex-valued* C^∞-functions on $z(U)$.

We can also define

$$\mathscr{A}^2_{\mathbb{C}}(X) := \mathscr{A}^2(X) \otimes_{\mathbb{R}} \mathbb{C}$$

Elements of $\mathscr{A}^2_{\mathbb{C}}(X)$ are called complex 2-forms and in local holomorphic co-ordinates these can written in the form $f(z)dz \wedge d\bar{z}$ with f a C^∞ complex-valued function.

Recall, we had earlier defined holomorphic 1-forms. These can be identified with complex 1-forms such that in the above representation, if f is holomorphic on $z(U)$ and $g \equiv 0$ on $z(U)$.

There are two complex vector fields associated with our given co-ordinate chart $z\colon U \to \mathbb{C}$, namely

$$\frac{\partial}{\partial z} := \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right]$$

and

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right].$$

Finally, it makes sense to talk about *holomorphic vector fields on a Riemann surface* X. These are vector fields $\mathbf{v}$ on X such that for every local holomorphic co-ordinate z, $\mathbf{v}$ can be written as $f\partial/\partial z$ with f holomorphic in the co-ordinate chart of z. The last problem shows that if $\mathbf{v}$ is a holomorphic vector field on X if we can find an atlas on X of holomorphic co-ordinate charts such that on each member of (U, z) of the atlas, $\mathbf{v}$ is of the form $f\partial/\partial z$ with f holomorphic.

At this point it is good to remember that T_X and $T_X \otimes_{\mathbb{R}} \mathbb{C}$ are naturally complex manifolds of dimension 2 and 3 respectively. It therefore makes sense to talk about holomorphic maps to and from these spaces from other complex manifolds (notably X).

(33) Show that a C^∞-function $f\colon U \to \mathbb{C}$ is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

[Remark: One can define these operators for a larger class of functions, example any function f for which the total $\mathbb{R}$-derivative exists as a matrix valued function on U. The statement remains true for f which is C^1.]

(34) (a) Show that if $f\colon U \to \mathbb{C}$ is a smooth function, then in $\mathscr{A}^1_{\mathbb{C}}(U)$ one has

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

[Here, df is defined as $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ or, more canonically, by the formula $df(\mathbf{v}) = \mathbf{v}(f)$.]

(b) Show that if $f\colon X \to \mathbb{C}$ is a C^∞-function, we have a well defined 1-forms ∂f and $\overline{\partial} f$ on X such that on a local holomorphic coordinate chart (U, z),

$$\partial f|_U = \frac{\partial f(z)}{\partial z}\, dz$$

and

$$\overline{\partial} f|_U = \frac{\partial f(z)}{\partial \overline{z}}\, d\overline{z}$$

(35) If $z^*\colon U \to \mathbb{C}$ is another holomorphic co-ordinate chart on U, show that

$$\frac{\partial}{\partial z^*} = \frac{dz}{dz^*}\frac{\partial}{\partial z}.$$

Deduce that if $\mathbf{v} = f\frac{\partial}{\partial z}$ is a complex vector field on U with $f\colon U \to \mathbb{C}$ holomorphic, then $\mathbf{v} = g\frac{\partial}{\partial z^*}$ with $g\colon U \to \mathbb{C}$ also holomorphic. [You have to show that if $\mathbf{v} = g\frac{\partial}{\partial z^*} + h\frac{\partial}{\partial \bar{z}^*}$ is the representation of $\mathbf{v}$ in terms of the chart z^*, then $h = 0$ on U and g is holomorphic.]

(36) Show that there is a decomposition of $\mathscr{T}_{\mathbb{C}}(X)$ as

$$(\dagger) \qquad \mathscr{T}_{\mathbb{C}}(X) = \mathscr{T}^{1,0}(X) \bigoplus \mathscr{T}^{0,1}(X)$$

where $\mathbf{v} \in \mathscr{T}^{1,0}(X)$ if the vector field $\mathbf{v}\colon X \to T(X) \otimes \mathbb{C}$ is locally of the form $f(z)\partial/\partial z$ on a holomorphic chart (U, z), with $f\colon U \to \mathbb{C}$ a C^∞-function on U, and $\mathbf{v}$ is in $\mathscr{T}^{0,1}$ if on this chart it can be expressed as locally as $f(z)\partial/\partial \bar{z}$ for f a C^∞-function on U. Is the decomposition (†) the same as the decomposition (*)?

(37) Suppose (U, z) is a holomorphic co-ordinate chart of X and $z = x + iy$ is the decomposition of the co-ordinate z into its real and imaginary parts.
   (a) Suppose u and v real-valued C^∞- functions on U. Consider the local map of vector fields $\mathbf{v} = u\partial/\partial x + v\partial/\partial y \mapsto (u + iv)\partial/\partial z = \mathbf{w}(\mathbf{v}, z)$. Show that $\mathbf{w}(\mathbf{v}, z)$ is independent of z. In other words, if $z^*\colon U \to \mathbb{C}$ is another holomorphic co-ordinate on U, and $z = x^* + iy^*$ its decomposition into real and imaginary parts, then show that $\mathbf{w}(\mathbf{v}, z) = \mathbf{w}(\mathbf{v}, z^*)$.
   (b) Show that there is a natural *global* isomorphism
   $$\mathbf{w}\colon \mathscr{T}(X) \xrightarrow{\sim} \mathscr{T}^{1,0}(X)$$
   which on holomorphic co-ordinate charts (U, z) restricts to $\mathbf{w}(-, z)$.

(38) Show that there is a decomposition

$$\mathscr{A}^1_{\mathbb{C}}(X) = \mathscr{A}^{1,0}(X) \bigoplus \mathscr{A}^{0,1}(X)$$

where in local holomorphic coordinates, elements of $\mathscr{A}^{1,0}(U)$ look like $f(z)dz$ and those of $\mathscr{A}^{0,1}(U)$ look like $f(z)d\bar{z}$, with $f(z)$ a C^∞ complex valued function on U.

(39) Show that holomorphic vector fields $\mathbf{v}\colon X \to T_X \otimes_{\mathbb{R}} \mathbb{C}$ are holomorphic maps. Show also that such a $\mathbf{v}$ corresponds to a holomorphic section of $T_X \to X$.

(40) Let $\omega \in \mathscr{A}^1_{\mathbb{C}}(X)$. Suppose on a holomorphic coordinate chart (U, z) we have

$$\omega|_U = P(z)dz + Q(z)d\bar{z}.$$

Define

$$\partial\,\omega|_U = \frac{\partial Q}{\partial z}(z) dz \wedge d\bar{z}$$

and

$$\begin{aligned}\overline{\partial}\,\omega|_U &= \frac{\partial P}{\partial \bar{z}}(z) d\bar{z} \wedge dz \\ &= -\frac{\partial P}{\partial \bar{z}}(z) dz \wedge d\bar{z}.\end{aligned}$$

Show that $\partial\,\omega|_U$ and $\overline{\partial}\,\omega|_U$ do not depend on the holomorphic co-ordinate z on U. Conclude that we have maps

$$\partial\colon \mathscr{A}^1_{\mathbb{C}}(X) \to \mathscr{A}^2_{\mathbb{C}}(X)$$

and

$$\overline{\partial}\colon \mathscr{A}^1_{\mathbb{C}}(X) \to \mathscr{A}^2_{\mathbb{C}}(X).$$

(41) Show that $\omega \in \mathscr{A}^{1,0}(X)$ is a holomorphic 1-form if and only if $\overline{\partial}\,\omega = 0$.

Elliptic curves and addition formulas

In what follows we fix a lattice $\mathbf{\Lambda}$ in $\mathbb{C}$. For definiteness,

$$\mathbf{\Lambda} = \mathbb{Z}\omega_1 \bigoplus \mathbb{Z}\omega_2.$$

The symbol $\wp$ will denote the Weierstrass $\wp$-function with respect to $\mathbf{\Lambda}$. The complex numbers e_1, e_2, and e_3 have their usual meaning with respect to the above data.

The symbol E will denote the affine plane curve

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3)$$

and $\overline{E}$ will denote its projective completion in $\mathbb{P}^2$. The addition in $\overline{E}$ induced by the canonical isomorphism $\mathbb{C}/BL \xrightarrow{\sim} \overline{E}$ will be denoted $\oplus$ and subtraction by $\ominus$.

(42) For k a non-negative integer, define $G_k = \sum_{\omega \in \mathbf{\Lambda}\setminus 0}(1/\omega^{2k})$. Show that

$$\wp'^2 = 4\wp^3 - 60G_2\wp - 140G_3.$$

(43) Show
 (a) $\ominus[\alpha,\,\beta,\,\gamma] = [\alpha, -\beta,\,\gamma]$ for every $[\alpha,\,\beta,\,\gamma] \in \overline{E}$.
 (b) There exist polynomials $P_i(X,\,Y,\,Z;\,U,\,V,\,W)$, $i = 1,2,3$, homogeneous of degree 2 in X, Y, Z, and homogeneous of degree 2 in U, V, and W, such that

$$[\alpha,\,\beta,\,\gamma] \oplus [\lambda,\,\mu,\,\nu] =$$
$$[P_1(\alpha,\,\beta,\,\gamma;\,\lambda,\,\mu,\,\nu), P_2(\alpha,\,\beta,\,\gamma;\,\lambda,\,\mu,\,\nu), P_3(\alpha,\,\beta,\,\gamma;\,\lambda,\,\mu,\,\nu)].$$

(44) Let

$$\lambda = \frac{e_1 - e_3}{e_2 - e_3}$$

and

$$j = \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1-\lambda)^2}.$$

Prove *Salmon's theorem*, viz., two elliptic curves C and C' are isomorphic if and only if they have the same j. You may use the fact that every elliptic curve is of the form $\mathbb{C}/\mathbf{\Lambda}$, and you may use the $(\wp,\,\wp')$ parameterization.

(45) A flex point on a projective plane curve is defined to be a point where the tangent line is a point of triple contact with the curve. Show that $\overline{E}$ has nine flex points and that a line joining any two passes through a third.

(46) Show that $3n$ points $(\wp(u_i),\, \wp'(u_i))$, $1 \le i \le 3n$ lie on a curve of degree n if and only if $\sum_{i=1}^{n} u_i \in \mathbf{\Lambda}$.

(47) Show that the addition theorem for the $\wp$-function can be written as

$$\int_{\infty}^{a} \frac{dx}{\sqrt{q(x)}} + \int_{\infty}^{b} \frac{dx}{\sqrt{q(x)}} = \int_{\infty}^{-a-b+\frac{1}{4}\left(\frac{\sqrt{q(a)}-\sqrt{q(b)}}{a-b}\right)^2} \frac{dx}{\sqrt{q(x)}}$$

where $q(x) = 4(x-e_1)(x-e_2)(x-e_3)$. Give suitable interpretations for the integrals and for the resulting "equality".

(48) (Euler 1753) Prove

$$\int_0^u \frac{dt}{\sqrt{1-t^4}} + \int_0^v \frac{dt}{\sqrt{1-t^4}} = \int_0^w \frac{dt}{\sqrt{1-t^4}}$$

where

$$w = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1+u^2v^2}.$$

(49) (Fagnano 1718) Here is the result that began it all: Fagnano's result on *doubling the arc of the lemniscate*. Show that

$$\int_0^r \frac{dx}{\sqrt{1-x^4}} = 2\int_0^u \frac{dx}{\sqrt{1-x^4}}$$

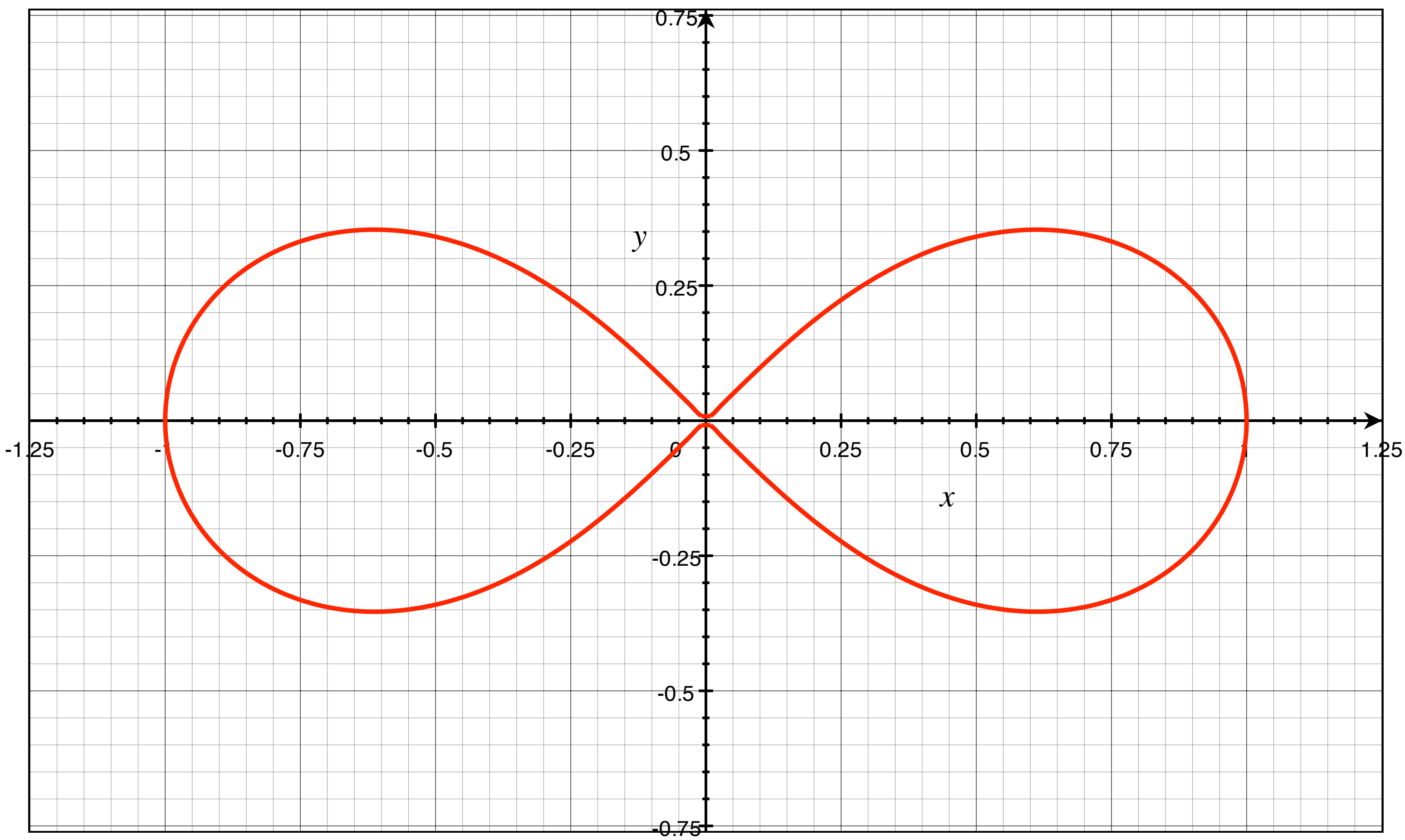
where

$$r = \frac{2u\sqrt{1-u^4}}{1+u^4}.$$

Here is some information being offered as background: The integral

$$\int_0^u \frac{dx}{\sqrt{1-x^4}}$$

is the arc length of the lemniscate $(x^2+y^2)^2 = x^2 - y^2$ calculated from the origin to a point on the lemniscate at a distance u from the origin. The lemniscate can also be described as the locus of a variable point Q whose distances d_1 and d_2 from two fixed point P_1 and P_2 are such that the product $d_1 d_2$ is a constant. The picture is included below (at worst see the next page).



0.75
0.5
0.25
-0.25
-0.5
-0.75
y
x
-1.25
-0.75
-0.5
-0.25
0
0.25
0.5
0.75
1.25

Vector Bundles, Cech cohomology

The notations and conventions are as in the document `paracompact.pdf` posted on the moodle site. Smooth and C^∞ are to be regarded as inter-changeable terms. Unless otherwise stated all spaces are smooth differentiable manifolds and all vector-bundles are smooth and complex valued.

(50) Let $E \to X$ and $F \to X$ be two rank n vector bundles, $\mathscr{U} = \{U_\alpha\}$ a common trivializing open cover for both of them, and $\{g_{\alpha\beta}\}$, $\{h_{\alpha\beta}\}$ transition maps for E and F respectively.

(a) If we have smooth maps $f_\alpha\colon U_\alpha \to GL_n(\mathbb{C})$, one for each index α, such that

$$g_{\alpha\beta}(x) f_\beta(x) = f_\alpha(x) h_{\alpha\beta}(x), \qquad x \in U_{\alpha\beta}$$

then show that E and F are isomorphic.

(b) Suppose E and F are holomorphic, what modifications would you make to the above statement to get a sufficient condition for E and F to be isomorphic as holomorphic bundles?

(51) Let X be *paracompact* and $E \to X$ a vector bundle. Let $\mathscr{U} = \{U_\alpha \,|\, \alpha \in I\}$ be a *locally finite* cover of X. Show that

$$0 \to \Gamma(X\, \mathscr{E}(E)) \xrightarrow{\imath} C^0(\mathscr{U}, \mathscr{E}(E)) \xrightarrow{\delta} C^1(\mathscr{U}, \mathscr{E}(E)) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^q(\mathscr{U}, \mathscr{E}(E)) \xrightarrow{\delta} \ldots$$

is exact.

(52) Let X be a topological space and $p \in X$ a point. Let $\widetilde{\mathbb{C}}_p$ be the sheaf given by

$$\widetilde{\mathbb{C}}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{otherwise} \end{cases}$$

with the restriction maps r_U^V being $1_{\mathbb{C}}$ if $p \in U$ and zero otherwise. Show that

$$\mathrm{H}^q(\mathscr{U}, \widetilde{\mathbb{C}}_p) = \begin{cases} \mathbb{C} & \text{if } q = 0 \\ 0 & \text{if } q \geq 1 \end{cases}$$

(53) Let X be a Riemann surface and $E \to X$ a holomorphic vector-bundle of rank n. Show that there is a well defined map

$$\overline{\partial}\colon \mathscr{E}(E) \to \mathscr{E}^{0,1}(E)$$

such that if (U, z) is a co-ordinate chart on which E is holomorphically trivial and $\mathscr{E}(E)|_U$ is identified with $\mathscr{E}^n|_U$ via this trivialization on U, then $\overline{\partial}$ is given on open subsets of U by

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial f_1}{\partial \overline{z}} d\overline{z} \\ \vdots \\ \frac{\partial f_n}{\partial \overline{z}} d\overline{z} \end{pmatrix}.$$

(54) Let X be a Riemann surface and $E \to X$ a holomorphic vector bundle.
  (a) Show that if $E = E^{1,0}$, then $\mathscr{E}^{0,1}(E) = \mathscr{E}_X^{1,1}$.
  (b) For every open set U in X, show that

$$\Gamma(U, E) = \ker\left(\mathscr{E}(E)(U) \xrightarrow{\overline{\partial}} \mathscr{E}^{0,1}(E)(U)\right).$$

  (c) For every open set U in X, show that

$$\Gamma(U, \Omega_X^1) = \ker\left(\mathscr{E}_X^{1,0}(U) \xrightarrow{\overline{\partial}} \mathscr{E}_X^{1,1}\right)(U).$$

In the problem that follows you may use the following result (which will be proved in class): *Suppose U is an open subset of $\mathbb{C}$ and $g\mathscr{U} \to \mathbb{C}$ is a smooth function. Then the differential equation*

$$\frac{\partial f}{\partial \overline{z}} = g$$

has a smooth solution in U.

(55) Let X be a Riemann surface and $E \to X$ a holomorphic vector bundle. For $q \geq 0$, define the *q-th cohomology with coefficients in* E, denoted $\mathrm{H}^q(X, E)$, to be the q-th cohomology of the complex

$$0 \to \Gamma(X, \mathscr{E}(E)) \xrightarrow{\overline{\partial}} \Gamma(X, \mathscr{E}^{0,1}(E)) \to 0.$$

  (a) Show that $\mathrm{H}^0(X, E) = \Gamma(X, E)$.
  (b) If X is paracompact and $\mathscr{U}$ is a locally finite open cover by holomorphic co-ordinate charts of X, show that there are natural isomorphisms

$$\mathrm{H}^q(\mathscr{U}, E) \xrightarrow{\sim} \mathrm{H}^q(X, E) \qquad q = 0, 1, \ldots.$$