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1. **The topological case**

For a topological space Z, P_Z will denote the category of presheaves on Z and Sh_Z the category of sheaves of abelian groups on Z.

1.1. **Inverse Image.** Let $f\colon X \to Y$ be a continuous map between topological spaces. The inverse image $f^{-1}\mathscr{G}$ of a sheaf $\mathscr{G}$ on Y is the sheaf on X which is obtained by sheafifying the presheaf

$$U \mapsto \varinjlim_{W} \mathscr{G}(W). \tag{1.1.1}$$

where the direct limit varies over open sets W of Y such that $U \subset f^{-1}(W)$. The restriction maps are the obvious ones. In this section we denote the presheaf defined by (1.1.1) by $f^{\#}\mathscr{G}$. The symbol $f^{\#}\mathscr{G}$ will be used for a different functor in the next section. By the universal property of sheafification, for each such $\mathscr{G}$ and each $\mathscr{F} \in \mathrm{Sh}_X$, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{P}_X}(f^{\#}\mathscr{G},\, \mathscr{F}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathrm{Sh}_X}(f^{-1}\mathscr{G},\, \mathscr{F}) \tag{1.1.2}$$

which is bifunctorial in $\mathscr{F}$ and $\mathscr{G}$ and such that when $\mathscr{F} = f^{-1}\mathscr{G}$ the sheafification map $\mathscr{G}^{\#} \to f^{-1}\mathscr{G}$ (which is an element in the group on the left side of (1.1.2)) maps to the identity element in $\mathrm{Hom}_{\mathrm{Sh}_X}(f^{-1}\mathscr{G},\, f^{-1}\mathscr{G})$ under (1.1.2).

1.2. **Adjointness.** For $\mathscr{F} \in \mathrm{Sh}_Y$ and $\mathscr{G} \in \mathrm{Sh}_X$, we claim that there is a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}_X}(f^{-1}\mathscr{G},\, \mathscr{F}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathrm{Sh}_Y}(\mathscr{G},\, f_*\mathscr{F}). \tag{1.2.1}$$

The map (1.2.1) is defined as follows. If $\varphi\colon f^{-1}\mathscr{G} \to \mathscr{F}$ is a map of sheaves, then, composing with the sheafification map $f^{\#}\mathscr{G} \to f^{-1}\mathscr{G}$, we get a map of presheaves $\bar{\varphi}\colon f^{\#}\mathscr{G} \to \mathscr{F}$. In fact, as a little thought shows, the map $\varphi \mapsto \bar{\varphi}$ is the inverse of the isomorphism (1.1.2). Let W be an open subset of Y. Then $(f^{\#}\mathscr{G})(f^{-1}(W)) = \mathscr{G}(W)$. Consider the composite

$$\mathscr{G}(W) = (f^{\#}\mathscr{G})(f^{-1}(W)) \xrightarrow{\bar{\varphi}(f^{-1}(W))} \mathscr{F}(f^{-1}(W)) = f_*\mathscr{F}(W).$$

It is easy to see this varies well with W, i.e., that as W varies, we get a map of sheaves $\mathscr{G} \to f_*\mathscr{F}$. We thus have a map

$$\mathrm{Hom}_{\mathrm{Sh}_X}(f^{-1}\mathscr{G},\, \mathscr{F}) \to \mathrm{Hom}_{\mathrm{Sh}_Y}(\mathscr{G},\, f_*\mathscr{F}). \tag{†}$$

To show it is an isomorphism, suppose we have $\psi \in \mathrm{Hom}_{\mathrm{Sh}_Y}(\mathscr{G},\, f_*\mathscr{F})$. Let U be an open set in X and W an open set in Y such that $f^{-1}(W) \supset U$. We have a map

$$\mathscr{G}(W) \longrightarrow \mathscr{F}(U) \tag{$*$}_W$$

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given by the composite

$$\mathscr{G}(W) \xrightarrow{\psi(W)} f_*\mathscr{F}(W) = \mathscr{F}(f^{-1}(W)) \longrightarrow \mathscr{F}(U).$$

One checks that if W' is an open subset of Y such that $f(U) \subset W' \subset W$ then the diagram

$$\begin{array}{ccc} \mathscr{G})(W) & & \\ {\scriptstyle \text{restriction}}\downarrow & \searrow^{(*)_W} & \\ \mathscr{G}(W') & \xrightarrow[(*)_{W'}]{} & \mathscr{F}(U) \end{array}$$

commutes. Taking direct limits we get a map

$$\widetilde{\psi}(U)\colon f^{\#}\mathscr{G}(U) \to \mathscr{F}(U)$$

and for V an open subset of X contained in U, one checks from the definitions that the following diagram commutes

$$\begin{array}{ccc} f^{\#}\mathscr{G}(U) & \xrightarrow{\widetilde{\psi}(U)} & \mathscr{F}(U) \\ {\scriptstyle \text{restriction}}\downarrow & & \downarrow{\scriptstyle \text{restriction}} \\ f^{\#}\mathscr{G}(V) & \xrightarrow[\widetilde{\psi}(V)]{} & \mathscr{F}(V), \end{array}$$

i.e., $\widetilde{\psi}\colon f^{\#}\mathscr{G} \to \mathscr{F}$ is a map of presheaves . We thus have a map

$$\mathrm{Hom}_{\mathrm{Sh}_Y}(\mathscr{G},\, f_*\mathscr{F}) \longrightarrow \mathrm{Hom}_{\mathrm{P}_X}(f^{\#}\mathscr{G},\, \mathscr{F})$$

given by $\psi \mapsto \widetilde{\psi}$. Applying (1.1.2) we therefore get

$$(\ddagger). \qquad \mathrm{Hom}_{\mathrm{Sh}_Y}(\mathscr{G},\, f_*\mathscr{F}) \longrightarrow \mathrm{Hom}_{\mathrm{Sh}_X}(f^{-1}\mathscr{G},\, \mathscr{F})$$

It is easy to see that (†) and (‡) are inverses of each other.

2. **The case of schemes**

2.1. Now suppose $f\colon X \to Y$ is a map of schemes. We claim that for an $\mathscr{O}_X$-module $\mathscr{F}$ and an $\mathscr{O}_Y$-module $\mathscr{G}$, we have a bifunctorial isomorphism

$$(2.1.1) \qquad \mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G},\, \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G},\, f_*\mathscr{F}).$$

We now change notation and write $f^{\#}\mathscr{G}$ for the presheaf

$$U \mapsto f^{-1}\mathscr{G}(U) \otimes_{f^{-1}\mathscr{O}_Y(U)} \mathscr{O}_X(U).$$

By definition $f^*\mathscr{G}$ is the sheafification of $f^{\#}\mathscr{G}$. Note that $f^{\#}\mathscr{G}$ is an $\mathscr{O}_X$-module. Recall that if $A \to B$ is a map of (commutative) rings, then

$$(*) \qquad \mathrm{Hom}_B(M \otimes_A B, N) \xrightarrow{\sim} \mathrm{Hom}_A(M, N).$$

Now suppose we have an $\mathscr{O}_X$-module map $f^{\#}\mathscr{G} \to \mathscr{F}$. For each open U in X a map $f^{-1}\mathscr{G}(U)\otimes_{f^{-1}\mathscr{O}_Y(U)}\mathscr{O}_X(U) \to \mathscr{F}(U)$, and hence (via $(*)$) a map $f^{-1}\mathscr{G}(U) \to \mathscr{F}(U)$. The maps $f^{-1}\mathscr{G}(U) \to \mathscr{F}(U)$ behave well with respect to the open sets U giving us a map of sheaves $f^{-1}\mathscr{G} \to \mathscr{F}$. (This map is easily seen to be a map of $f^{-1}\mathscr{O}_Y$ modules.) From the first section we know that we therefore have a map of sheaves

$\mathscr{G} \to f_*\mathscr{F}$. It is not hard to see that this is a map of $\mathscr{O}_Y$-modules, since it arises from a map of $f^{-1}\mathscr{O}_Y$-modules. We thus have a map:

$$(**) \qquad \mathrm{Hom}_{\mathscr{O}_X}(f^{\#}\mathscr{G},\, \mathscr{F}) \longrightarrow \mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{G},\, \mathscr{F}).$$

Conversely, given a map of $\mathscr{O}_Y$-modules $\psi\colon \mathscr{G} \to f_*\mathscr{F}$, we get from the first section a map $\widehat{\psi}\colon f^{-1}\mathscr{G} \to \mathscr{F}$ and this is seen easily to be a $f^{-1}\mathscr{O}_Y$-module map. From $(*)$, we have the identification

$$\mathrm{Hom}_{f^{-1}\mathscr{O}_Y(U)}(f^{-1}\mathscr{G}(U),\, \mathscr{F}(U)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_X(U)}(f^{-1}\mathscr{G}(U) \otimes_{f^{-1}\mathscr{O}_Y(U)} \mathscr{O}_X(U),\, \mathscr{F}(U)).$$

Thus the $f^{-1}\mathscr{O}_Y$-module map $\widehat{\psi}\colon f^{-1}\mathscr{G} \to \mathscr{F}$ gives rise to a $\mathscr{O}_X$-module map $f^{\#}\mathscr{G} \to \mathscr{F}$. In other words $(**)$ is an isomorphism, since the two processes are easily seen to be inverses of each other. Moreover, from the universal property of sheafification, we get an $\mathscr{O}_X$-module map $f^*\mathscr{G} \to \mathscr{F}$. This essentially establishes (2.1.1). For good book-keeping, here is the hierarchy of identifications:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G},\, \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_X}(f^{\#}\mathscr{G},\, \mathscr{F}) &\xrightarrow{\sim} \mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{G},\, \mathscr{F}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G},\, f_*\mathscr{F}). \end{aligned}$$

The first isomorphism above is from the universal property of sheafification (which one checks respects $\mathscr{O}_X$-structures). The second is $(**)$ which we have shown is an isomorphism. The third is from the topological case considered in section one, and noting that the adjunction isomorphism there is such that if $f^{-1}\mathscr{G} \to \mathscr{F}$ is a map of $f^{-1}\mathscr{O}_Y$-modules, then its adjoint, namely $\mathscr{G} \to f_*\mathscr{F}$, is a map of $\mathscr{O}_Y$-modules.

References

[EGA-III] A Grothendieck and J. Dieudonne, *Élements de Géométrie Algébrique* **III**, Publications Math. IHES **11**, Paris, 1961

[H] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977.