Semi-continuity

Let A be a noetherian ring (noetherian only needed sporadically, but it is convenient to make the blanket assumption). Let $C^\bullet$ be a flat bounded above complex, say

$$\longrightarrow C^{n-1} \longrightarrow C^n \longrightarrow \cdots \longrightarrow C^{N-1} \longrightarrow C^N \longrightarrow 0.$$

For $i \in \mathbb{Z}$ and $M \in \mathrm{Mod}_A$, set   The category of A-modules

$$T^i(M) = H^i(C^\bullet \otimes_A M).$$

Then T^i is a δ-functor, i.e., given a short exact seq of A-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0, \qquad (E)$$

we have a long exact sequence

$$\left\{ \begin{array}{l} \cdots \longrightarrow T^{i-1}(M'') \xrightarrow{\delta} T^i(M') \longrightarrow T^i(M) \longrightarrow T^i(M'') \xrightarrow{\delta} T^{i+1}(M') \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow \cdots \end{array} \right.$$

which is functorial in (E). This is because

$$0 \longrightarrow C^\bullet \otimes_A M' \longrightarrow C^\bullet \otimes_A M \longrightarrow C^\bullet \otimes_A M'' \longrightarrow 0 \qquad (C^\bullet \otimes E)$$

is an exact sequence of complexes (since each C^n is flat) and the correspondence $(E) \longmapsto (C^\bullet \otimes E)$ is functorial in (E).

Notations:

Fix $i \in \mathbb{Z}$ and $M \in \mathrm{Mod}_A$. Set

$$Z^i(M) = \ker(C^i \otimes_A M \longrightarrow C^{i+1} \otimes_A M)$$

$$B^i(M) = \mathrm{image}(C^{i-1} \otimes_A M \longrightarrow C^i \otimes_A M)$$

$$= \mathrm{coker}(Z^{i-1}(M) \longrightarrow C^i \otimes_A M)$$

$$W^i(M) = \mathrm{coker}(B^i(M) \longrightarrow C^i \otimes_A M).$$

We have the following exact sequences:

$$\left.\begin{array}{l} 0 \longrightarrow B^i(M) \longrightarrow Z^i(M) \longrightarrow T^i(M) \longrightarrow 0 \\[2em] 0 \longrightarrow T^i(M) \longrightarrow W^i(M) \longrightarrow B^{i+1}(M) \longrightarrow 0 \end{array}\right\} \quad i \in \mathbb{Z}.$$

We have natural maps (for $i \in \mathbb{Z}$)

$$\left.\begin{array}{l} Z^i(A) \otimes_A M \longrightarrow Z^i(M) \\ B^i(A) \otimes_A M \longrightarrow B^i(M) \end{array}\right\} \quad \text{see below for existence.}$$

whence a natural map

$$\varphi^i(M) : \quad T^i(A) \otimes_A M \longrightarrow T^i(M)$$

given by filling in the dotted arrow in the commutative diagram with exact rows:

$$\begin{tikzcd} & B^i(A) \otimes_A M \arrow[r] \arrow[d] & Z^i(A) \otimes_A M \arrow[r] \arrow[d] & T^i(A) \otimes_A M \arrow[r] \arrow[d, dashed, "\varphi^i"] & 0 \\ 0 \arrow[r] & B^i(M) \arrow[r] & Z^i(M) \arrow[r] & T^i(M) \arrow[r] & 0 \end{tikzcd}$$

The map $B^i(A) \otimes_A M \longrightarrow B^i(M)$ is defined as follows and is in fact surjective! Indeed we have an exact seq.

$$Z^{i-1}(A) \otimes_A M \longrightarrow C^{i-1} \otimes_A M \longrightarrow B^i(A) \otimes_A M \longrightarrow 0$$

which means $C^{i-1} \otimes_A M \longrightarrow B^i(A) \otimes_A M$ is surjective.

On the other hand the commutative diagram

$$\begin{tikzcd} C^{i-1} \arrow[rr, "\partial^{i-1}"] \arrow[dr, two heads] & & C^i \\ & B^i(A) \arrow[ur] & \end{tikzcd}$$

surjective

yields the comm. diag

$$\begin{tikzcd}
C^{i-1}\otimes_A M \arrow[rr, "\partial^{i-1}\otimes M"] \arrow[dr, "\text{surjective}"'] & & C^i\otimes_A M \\
& B^i(A)\otimes_A M \arrow[ur] &
\end{tikzcd}$$

It follows that the north-east pointing arrow has image equal to the image of $\partial^{i-1}\otimes M$, i.e., equal to $B^i(M)$. This defines $B^i(A)\otimes_A M \longrightarrow B^i(M)$ and this is clearly surjective.

The map

$$Z^i(A)\otimes_A M \longrightarrow Z^i(M)$$

is defined by filling in the dotted arrow in the CD with exact rows:

$$\begin{tikzcd}
Z^i(A)\otimes_A M \arrow[r] \arrow[d, dashed] & C^i\otimes_A M \arrow[r] \arrow[d, equal] & B^{i+1}(A)\otimes_A M \arrow[r] \arrow[d, two heads] & 0 \\
Z^i(M) \arrow[r] & C^i\otimes_A M \arrow[r] & B^{i+1}(M) \arrow[r] & 0
\end{tikzcd}$$

Finally note that $\operatorname{coker}(B^i(A)\otimes_A M \longrightarrow C^i\otimes_A M) = \operatorname{coker}(B^i(M)\longrightarrow C^i\otimes_A M)$ since $B^i(A)\otimes_A M \longrightarrow B^i(M)$ is surjective. This means

$$\boxed{W^i(A)\otimes_A M = W^i(M).} \longleftarrow \text{Important.}$$

<u>Note</u>: The above discussion on Z^i, B^i did not require flatness of $C^\bullet$. Nonetheless, let us persist with the assumption. The naturality of Z^i, B^i, W^i (i.e., their functoriality) is

quite obvious.

Lemma: Suppose $f : P \longrightarrow V$ is an A-module map between finite projective modules. Then the functor $G : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_A$ given by $G(M) = \ker(f \otimes M)$ is of the form $G = \mathrm{Hom}_A(Q, -)$ for a unique (up to isomorphism) A-module Q. In particular G is left exact.

Proof: Let $Q = \mathrm{coker}\left(V^\vee \xrightarrow{f^\vee} P^\vee\right)$, i.e., the cokernel of the transpose of f. Then we have a CD with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(Q, M) & \longrightarrow & \mathrm{Hom}_A(P^\vee, M) & \longrightarrow & \mathrm{Hom}_A(V^\vee, M) \\ & & & & \| & & \| \\ 0 & \longrightarrow & G(M) & \longrightarrow & P \otimes_A M & \longrightarrow & V \otimes_A M \end{array}$$

Thus $G(M) \simeq \mathrm{Hom}_A(Q, M)$. Uniqueness follows since Q represents G. covariant functor

q.e.d.

We now examine conditions under which T^i is left exact, right exact or exact.

Proposition 1: Fix $i \in \mathbb{Z}$. The following are equivalent.

(i) T^i is left exact

(ii) $W^i(A)$ is flat.

equal to f.g. A-modules

Moreover if $C^\bullet$ consists of finite A-modules then (i) and (ii) are equivalent to

(iii) There exists a finite A-module Q such that

$$T^i = \mathrm{Hom}_A(Q, -).$$

This Q is unique up to isomorphism.

Proof:

Suppose $M \hookrightarrow N$ is an injective map of complexes. We have a CD with exactness as indicated:

$$
\begin{tikzcd}
 & & W^i(M) \arrow[d, phantom, "\parallel"] & 0 \arrow[d] & \\
0 \arrow[r] & T^i(M) \arrow[r] \arrow[d] & W^i(A)\otimes_A M \arrow[r] \arrow[d] & C^{i+1}\otimes_A M \arrow[d] & \text{(exact)} \\
0 \arrow[r] & T^i(N) \arrow[r] & W^i(A)\otimes_A N \arrow[r] & C^{i+1}\otimes_A N & \text{(exact)} \\
 & & W^i(N) \arrow[u, phantom, "\parallel"] & \text{(exact)} &
\end{tikzcd}
$$

The column on the right is exact since C^{i+1} is flat. This is the first time we've used this hypothesis. It follows from the CD that the downward arrow on the left is injective if and only if the downward arrow on the right is injective. Thus (i) $\iff$ (ii).

Now suppose $C^\bullet$ consists of finite A-modules. Suppose (ii) is true. Then, $W^i(A)$ being finite, it must be projective. So is C^{i+1}. Setting $W^i(A) = P$, $C^{i+1} = V$, and f equal to the map $W^i(A) \to C^{i+1}$ and applying the previous lemma, we get (ii). Finally (iii) clearly implies (i). q.e.d.

Recall, for $M \in \mathrm{Mod}_A$, $\varphi^i(M)$ is the map $T^i(A)\otimes_A M \longrightarrow T^i(M)$. Note that if the natural transformation φ^i is a functorial isomorphism then T^i is necessarily right exact since $T^i(A)\otimes_A -$ is. This is in fact an "if and only if" condition as seen below.

Proposition 2: Fix $i \in \mathbb{Z}$. The following are equivalent

(i) T^i is right exact.

(ii) φ^i is an isomorphism

(iii) $\varphi^i(M)$ is surjective for every $M \in \mathrm{Mod}_A$

If A is a local ring with residue field k, and $T^j(A)$ are finitely gen'd over A $\forall j$, then (i), (ii) and (iii) above are equivalent to

(iv) $\varphi^i(k): T^i(A) \otimes_A k \longrightarrow T^i(k)$ is surjective.

Proof:

Suppose T^i is right exact. Let M be a f.g. A-module. We have a presentation $(r, s < \infty)$

$$A^r \longrightarrow A^s \longrightarrow M \longrightarrow 0 \qquad \text{(exact)}$$

and hence a commutative diagram with exact rows,

$$\begin{tikzcd} T^i(A)\otimes_A A^r \arrow[r] \arrow[d, "\varphi^i(A^r)"', "\wr"] & T^i(A)\otimes_A A^s \arrow[r] \arrow[d, "\varphi^i(A^s)"', "\wr"] & T^i(A)\otimes_A M \arrow[r] \arrow[d, "\varphi^i(M)"] & 0 \quad \text{(exact)} \\ T^i(A^r) \arrow[r] & T^i(A^s) \arrow[r] & T^i(M) \arrow[r] & 0 \quad \text{(exact)} \end{tikzcd}$$

and it follows that $\varphi^i(M)$ is an isomorphism whenever M is f.g. Since every module is the direct limit of f.g. modules and since tensor products and cohomology (in particular T^i) commute with direct limits, it follows that $\varphi^i(M)$ is an isomorphism $\forall M \in \mathrm{Mod}_A$. Thus (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (iii) is obvious.

Suppose $\varphi^i(M)$ is surjective for all $M \in \mathrm{Mod}_A$. Let M be f.g. Then we have a presentation (as before)

$$A^r \longrightarrow A^s \longrightarrow M \longrightarrow 0 \qquad \text{(exact)}$$

whence a commutative diagram with exact rows:

$$\begin{array}{ccccccc} T^i(A)\otimes_A A^r & \longrightarrow & T^i(A)\otimes_A A^s & \longrightarrow & T^i(A)\otimes_A M & \longrightarrow 0 & \text{(exact)} \\ \varphi^i(A^r)\downarrow\wr & & \varphi^i(A^s)\downarrow\wr & & \downarrow \varphi^i(M) & & \\ T^i(A^r) & \longrightarrow & T^i(A^s) & \longrightarrow & T^i(M) & & \text{(exact).} \end{array}$$

Since $\varphi^i(M)$ is surjective, therefore $T^i(A^s)\longrightarrow T^i(M)$ is surjective, and it follows that $\varphi^i(M)$ is an isomorphism by the 3-lemma (or uniqueness of cokernels). By the now familiar direct limit argument, it follows that $\varphi^i(M)$ is an isomorphism. $\forall$ M (f.g. or not). Thus (ii) $\Longrightarrow$ (iii).

Clearly (iii) $\Rightarrow$ (i). Thus (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Now suppose A is local with residue field k and $T^i(A)$ is f.g. clearly (iii) $\Rightarrow$ (iv). By Lemma 24 of the notes on homological algebra ["Some homological algebra (mapping cones etc)"], we see that (iv) $\Rightarrow$ (ii). q.e.d.

Proposition 3: Suppose φ^i is an isomorphism. Then φ^{i-1} is an isomorphism if and only if $T^i(A)$ is flat. In particular, if $T^i(A)$ is f.g. then φ^{i-1} is an isomorphism if and only if $T^i(A)$ is a projective A-module.

Proof:

The map φ^{i-1} is an isomorphism if and only if T^{i-1} is right exact, i.e. if and only if T^i is left exact. Now φ^i is an isomorphism, and hence T^i is right exact, and hence exact. Thus $M\mapsto T^i(A)\otimes_A M$ is exact (since $T^i(A)\otimes - \cong T^i$). This means $T^i(A)$ is flat. q.e.d.

Local global principles: We assume $C^\bullet$ has finite cohomology.

From Lemma 2 and Lemma 3 in the note on homological alg ("Some homological algebra ...") we may assume $C^\bullet$ is a complex of f.g. A-modules.

Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Let $T^i_{\mathfrak{p}}$ be the restriction of T^i to $\mathrm{Mod}_{A_\mathfrak{p}}$. Note that $T^i_\mathfrak{p} = H^i(C^\bullet_\mathfrak{p} \otimes_{A_\mathfrak{p}} -)$ since $C^\bullet \otimes_A M = C^\bullet_\mathfrak{p} \otimes_{A_\mathfrak{p}} M$ for any $M \in \mathrm{Mod}_{A_\mathfrak{p}}$. Also note that

$$T^i(M)_\mathfrak{p} = T^i_\mathfrak{p}(M_\mathfrak{p}) \qquad \forall\, M \in \mathrm{Mod}_A$$

since localisation commutes with cohomology.

Remark: The above considerations show that if $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, the residue field at $\mathfrak{p}$, then $T^i(k(\mathfrak{p}))$ is a finite dimensional vector space.

Definition: We say T^i is right exact (resp. left exact, resp. exact) at $\mathfrak{p}$ if $T^i_\mathfrak{p}$ is right exact (resp. left exact, resp. exact).

Note that if $f \in A$, then we have the identifications

$$T^i(M)_f = T^i(M_f) = H^i(C^\bullet_f \otimes_{A_f} M_f) = H^i(C^\bullet_f \otimes_A M),$$

since cohomology commutes with localisation and since

$$C^\bullet \otimes_A M_f = C^\bullet \otimes_A (A_f \otimes_A M) = C^\bullet_f \otimes_A M = C^\bullet_f \otimes_{A_f} M_f \text{ etc.}$$

Definition: We say T^i is right exact (resp. left exact, resp. exact) in a neighbourhood of $\mathfrak{p}$ if $\exists\, f \in A$, $f \notin \mathfrak{p}$ such that $M \longmapsto T^i(M)_f$ is right exact (resp. left exact, resp. exact).

Note that $M \longmapsto T^i(M)_f$ is right, left, or exact if and only if $N \longmapsto H^i(C_f^\bullet \otimes_{A_f} N)$ is right, left or exact on Mod_{A_f}. This more or less follows from the fact that $M \longrightarrow M_f$ is exact.

There is an obvious sheaf theoretic translation of all of the above, since $f \notin \mathfrak{p} \iff \mathrm{Spec}(A_f)$ is an open nbhd of $\mathfrak{p}$.

Proposition 4: T^i is right exact (resp. left exact, resp. exact) at $\mathfrak{p} \in \mathrm{Spec}\, A$ if and only if it is so in a neighborhood of $\mathfrak{p}$.

Proof: This is obvious. //

Proposition 5: Suppose $T^i(A)$ is f.g. $\forall i \in \mathbb{Z}$. Then for each $i \in \mathbb{Z}$, and each $\mathfrak{p} \in \mathrm{Spec}\, A$, the map

$$\mathfrak{p} \longmapsto \dim_{k(\mathfrak{p})} T^i(k(\mathfrak{p}))$$

is upper semi-continuous. Here as before, $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.

Proof:

For each $M \in \mathrm{Mod}_A$ we have an exact sequence

$$0 \longrightarrow T^i(M) \longrightarrow \underset{\overset{\|}{W^i(M)}}{W^i(A) \otimes_A M} \longrightarrow C^{i+1} \otimes_A M \longrightarrow \underset{\overset{\|}{W^{i+1}(M)}}{W^{i+1}(A) \otimes_A M} \longrightarrow 0$$

Set $M = k(\mathfrak{p})$. We get

$$\dim_{k(\mathfrak{p})} T^i(k(\mathfrak{p})) = \dim_{k(\mathfrak{p})} W^i(k(\mathfrak{p})) + \dim_{k(\mathfrak{p})} W^{i+1}(k(\mathfrak{p})) - \dim_{k(\mathfrak{p})} C^{i+1} \otimes_A k(\mathfrak{p}).$$

Since $\mathfrak{p} \longmapsto \dim_{k(\mathfrak{p})} C^{i+1} \otimes_A k(\mathfrak{p})$ is constant on connected components, and since $\mathfrak{p} \longmapsto \dim_{k(\mathfrak{p})} W^j(A) \otimes_A k(\mathfrak{p})$ is upper-semicontinuous $\forall j \in \mathbb{Z}$, we are done. q.e.d.

Semi-continuity in Algebraic Geometry

In this section

$$f: X \longrightarrow Y = \operatorname{Spec} A$$

is a proper map and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module which is flat over Y. Since f is proper and A noetherian, X is necessarily noetherian. In particular it has a finite open cover $\mathcal{U} = \{U_0, \ldots, U_N\}$ with each U_j an affine open subscheme of X. Further since X is separated, $U_{i_0 \ldots i_n} := U_{i_0} \cap \ldots \cap U_{i_n}$ is affine. Since $\mathcal{F}(U_{i_0 \ldots i_n})$ is flat over A, the Čech complex

$$C^\bullet := \check{C}^\bullet(\mathcal{U}, \mathcal{F})$$

is a bounded above flat complex. Moreover for $M \in \mathrm{Mod}_A$

$$C^\bullet \otimes_A M = \check{C}^\bullet(\mathcal{U}, \mathcal{F} \otimes_{\mathcal{O}_X} f^* \tilde{M}).$$

Since the $U_{i_0 \ldots i_n}$ are affine it follows that

$$H^i(C^\bullet \otimes_A M) \xrightarrow{\sim} H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} f^* \tilde{M}).$$

Finally, since f is proper and $\mathcal{F}$ is coherent, the right hand side of the above isomorphism is a f.g. A-module whenever M is f.g., and hence so is the left side. In particular, taking $M = A$, $C^\bullet$ has f.g. cohomology.

In summary $C^\bullet$ is a bounded above flat complex with finitely generated cohomology.

Let T^i, W^i, Z^i, B^i etc of the previous sections be computed with this $C^\bullet$. Note that

$$T^i \cong H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} f^* (-)^\sim).$$

Some notations: For $y \in Y$, let $k(y)$ be the residue field at y, $X_y := X \times_Y \operatorname{Spec} k(y)$, and $j_y : X_y \longrightarrow X$ the natural base change map. Let

Not the stalk of $\mathcal{F}$ at y! Since $y \in Y$, the stalk at y of $\mathcal{F}$ does not make sense $\longrightarrow$ $\mathcal{F}_y = j_y^* \mathcal{F}$

Proposition 1 gives [H, pp. 284-285, Prop. 12.4], Proposition 2 gives [H, p.286, Prop. 12.5]. Proposition 5 gives [H, p.288 Prop. 12.8], Proposition 4 gives [H, p.289, Prop. 12.10] and the cohomology and base change theorem [H, pp. 290-291, Thm 12.11] follows from Propositions 2, 3, 4 and 5.