IMPORTANT LEMMA

The following is a very useful Lemma due to Nitsure. The proof only needs Nakayama, and the following trivial observation, namely, given any complex $C^\bullet$ of modules over a ring A, then for any A-module M, the natural map $B^i(C^\bullet)\otimes_A M \to B^i(C^\bullet \otimes_A M)$ is surjective for every i. Here, as is usual $B^i(C^\bullet)$ is the A-module of i-co-boundaries of $C^\bullet$, and $Z^i(C^\bullet)$ will stand for the A-module of i-co-cycles.

Lemma 1. [Nitsure] *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $C^\bullet$ be the complex*

$$0 \to C^0 \to C^1 \to C^2 \to 0$$

consisting of finite A-modules with C^1 and C^2 free over A.

(1) *Suppose the natural map*

$$H^1(C^\bullet) \otimes_A k \to H^1(C^\bullet \otimes_A k)$$

is surjective. Then

(a) *$Z^1(C^\bullet)$ is a direct summand of C^1 and $B^2(C^\bullet)$ is a direct summand of C^2.*

(b) *For any A-module M, the natural map*

$$H^1(C^\bullet) \otimes_A M \to H^1(C^\bullet \otimes_A M)$$

is an isomorphism.

(2) *If $H^1(C^\bullet \otimes_A k) = 0$ then $H^1(C^\bullet) = 0$.*

Proof. We will first prove (1). Note that (b) follows from (a) because according to (a) the exact sequences

$$0 \to Z^1(C^\bullet) \to C^1 \to B^2(C^\bullet) \to 0 \tag{i}$$

and

$$0 \to B^2(C^\bullet) \to C^2 \to C^2/B^2(C^\bullet) \to 0 \tag{ii}$$

are split, and hence remain exact on tensoring with M. It is then easy to see that $Z^1(C^\bullet) \otimes M \to Z^1(C^\bullet \otimes M)$ and $B^2(C^\bullet) \otimes M \to B^2(C^\bullet \otimes M)$ are isomorphisms, and hence $H^1(C^\bullet) \otimes M \to H^1(C^\bullet \otimes M)$ is an isomorphism. To prove (a) (of (1)) it is enough to prove that $B^2(C^\bullet)$ is a direct summand of C^2. Indeed if it is, then $B^2(C^\bullet)$ is projective, whence the exact sequence (i) above splits, which in turn implies that $Z^1(C^\bullet)$ is direct summand of C^1. Thus in order to prove (1) it is enough to prove that $B^2(C^\bullet)$ is a direct summand of C^2. To that end consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & B^1(C^\bullet)\otimes_A k & \longrightarrow & Z^1(C^\bullet)\otimes_A k & \longrightarrow & H^1(C^\bullet)\otimes_A k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^1(C^\bullet\otimes_A k) & \longrightarrow & Z^1(C^\bullet\otimes_A k) & \longrightarrow & H^1(C^\bullet\otimes_A k) & \longrightarrow & 0
\end{array}$$

The south pointing arrow on the left is always surjective. By our hypothesis the south pointing arrow on the right is surjective. Usual arguments (snake lemma) then show that the middle downward arrow is surjective. Consider the commutative diagram

$$\begin{array}{ccccc} Z^1(C^\bullet) & \twoheadrightarrow & Z^1(C^\bullet) \otimes_A k & \twoheadrightarrow & Z^1(C^\bullet \otimes_A k) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ C^1 & \longrightarrow & C^1 \otimes_A k & = & C^1 \otimes_A k \end{array}$$

All the horizontal arrows are surjective and the leftmost downward arrow and the rightmost downward arrow are inclusions.

Pick a basis $\overline{u}_{,} \ldots, \overline{u}_p$ of the k-vector space $Z^1(C^\bullet \otimes_A k)$. These are linearly independent elements of $C^1 \otimes_A k$ and can be extended to a basis $\overline{u}_1, \ldots, \overline{u}_p, \overline{w}_1, \ldots, \overline{w}_r$ of $C^1 \otimes_A k$. Since the horizontal arrows on the top row are surjective, the elements $\overline{u}_{,} \ldots, \overline{u}_p$ can be lifted to elements $u_1, \ldots, u_p \in Z^1(C^\bullet)$. Similarly, $\overline{w}_1, \ldots, \overline{w}_r$ can be lifted to elements $w_1, \ldots, w_r$ in C^1. Standard Nakayama arguments show that $u_1, \ldots, u_p, w_1, \ldots, w_r$ forms a free basis for the free A-module C^1, with $u_1, \ldots, u_p \in Z^1(C^\bullet)$.

Clearly, if d denotes the coboundary map of $C^\bullet$, and of $C^\bullet \otimes_A k$, $dw_1, \ldots, dw_r$ generate $B^2(C^\bullet)$, and $d\overline{w}_1, \ldots, d\overline{w}_r$ forms a basis for $B^2(C^\bullet \otimes_A k)$ (the latter assertions follows from the fact that $\overline{u}_1, \ldots, \overline{u}_p, \overline{w}_1, \ldots, \overline{w}_r$ forms a basis for $C^1 \otimes_A k$ and $\{\overline{u}_j\}_j$ forms a basis for $Z^1(C^\bullet \otimes_A k)$). Consider the composite

$$B^2(C^\bullet) \twoheadrightarrow B^2(C^\bullet) \otimes_A k \to B^2(C^\bullet \otimes_A k).$$

The composite is surjective. The elements $d\overline{w}_1, \ldots, d\overline{w}_r$ can be extended to a basis of $C^2 \otimes_A k$ and by Nakayama any set of preimages in C^2 of this basis will form a free basis of C^2 over A. As a consequence $dw_1, \ldots, dw_r$ form a free basis of $B^2(C^\bullet)$. This proves (1).

Part (2) is an immediate consequence. Indeed, if $H^1(C^\bullet \otimes_A k)$ is zero, then the map $H^1(C^\bullet) \otimes_A k \to H^1(C^\bullet \otimes_A k)$ is surjective, whence by what we have proved, an isomorphism. Nakayama gives the rest.

□