DOUBLE COMPLEXES

We work throughout in an abelian category $\mathscr{A}$ in which *countable direct sums* exist.

There are two related notions of a double complex. We give both versions below. The second (i.e. what we call an *anti-commuting* double complex below) is what you will often find in the older literature—and amongst non-algebraic-geometers. The first version (which we simple call a double complex, or sometimes a *standard double complex*) is the version given by Grothendieck in EGA, and is what most algebraic geometers and commutative algebraists are used to. The difference is one of convention.

Standard Double Complexes. A *double complex* in $\mathscr{A}$, or sometimes in our class, a *standard double complex* in $\mathscr{A}$, consists of data $A^{\bullet\bullet} = (A, \partial_1, \partial_2)$, where

$$A = (A^{p,q})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$$

is a family of objects in $\mathscr{A}$, and

$$\partial_1 = (\partial_1^{p,q})_{(p,q)\in\mathbb{Z}} \qquad \partial_2 = (\partial_2^{p,q})_{(p,q)\in\mathbb{Z}}$$

are two families of morphisms

$$\partial_1^{p,q}\colon A^{p,q} \to A^{p+1,q} \qquad \partial_2^{p,q}\colon A^{p,q} \to A^{p,q+1}$$

such that

$$\partial_1\partial_1 = 0 \qquad \partial_2\partial_2 = 0 \qquad \partial_1\partial_2 = \partial_2\partial_1.$$

We often suppress the superscripts p, q when these are either immaterial or easily deducible from the context. Thus, e.g., we write ∂_2 for $\partial_2^{p,q}$. The maps ∂_1 and ∂_2 will be called partial coboundaries, and when we wish to be more specific, they will be called *horizontal* and *vertical* (partial) coboundaries respectively. The data fits into a commutative diagram, whose rows and columns are complexes.

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \cdots & & \vdots & & \vdots & & \\
 & & {\scriptstyle\partial_2}\uparrow & & \vdots & & {\scriptstyle\partial_2}\uparrow & & {\scriptstyle\partial_2}\uparrow & & \\
\cdots & \xrightarrow{\partial_1} & A^{0,q+1} & \xrightarrow{\partial_1} & \cdots & \xrightarrow{\partial_1} & A^{p,q+1} & \xrightarrow{\partial_1} & A^{p+1,q+1} & \xrightarrow{\partial_1} & \cdots \\
 & & {\scriptstyle\partial_2}\uparrow & & \vdots & & {\scriptstyle\partial_2}\uparrow & & {\scriptstyle\partial_2}\uparrow & & \\
\cdots & \xrightarrow{\partial_1} & A^{0,q} & \xrightarrow{\partial_1} & \cdots & \xrightarrow{\partial_1} & A^{p,q} & \xrightarrow{\partial_1} & A^{p+1,q} & \xrightarrow{\partial_1} & \cdots \\
 & & {\scriptstyle\partial_2}\uparrow & & \vdots & & {\scriptstyle\partial_2}\uparrow & & {\scriptstyle\partial_2}\uparrow & & \\
 & & \vdots & & \cdots & & \vdots & & \vdots & &
\end{array}
$$

Next consider the direct sum[1]

$$\mathrm{Tot}^n A^{\bullet\bullet} := \bigoplus_{p+q=n} A^{p,q}.$$

Define

$$\partial^n \colon \mathrm{Tot}^n A^{\bullet\bullet} \to \mathrm{Tot}^{n+1} A^{\bullet\bullet}$$

by the formula

$$\partial^n = \sum_{p+q=n} \{\partial_1^{p,q} + (-1)^p \partial_2^{p,q}\}.$$

The map within "curly brackets" can be regarded as a map $A^{p,q} \to \mathrm{Tot}^{n+1} A^{\bullet\bullet}$, taking values in the subobject $A^{p+1,q} \oplus A^{p,q+1}$ of $\mathrm{Tot}^{n+1} A^{\bullet\bullet}$, whence by the definition of direct sum, the map ∂^n makes sense.

Evidently

$$\partial^{n+1} \circ \partial^n = 0$$

for every $n \in \mathbb{Z}$ by the relations given between ∂_1 and ∂_2. We have therefore a complex $(\mathrm{Tot}^\bullet A^{\bullet\bullet}, \partial)$, called the *total complex* associated to the double complex $A^{\bullet\bullet}$.

A morphism of double complexes $\varphi\colon A^{\bullet\bullet} \to B^{\bullet\bullet}$ is (of course) a family of maps $f^{p,q}\colon A^{p,q} \to B^{p,q}$, one for each ordered pair of integers (p,q), which commute with vertical and horizontal coboundaries. This naturally induces a map of complexes $\mathrm{Tot}\, f\colon \mathrm{Tot}^\bullet A^{\bullet\bullet} \to \mathrm{Tot}^\bullet B^{\bullet\bullet}$

Anti-commutative double complexes. In much of the pre-Grothendieck literature, double complexes mean a variant of our standard double complexes. The only difference is that the grids in the diagram on the last page anti-commute rather than commute. In greater detail, for this course, data of the form $K^{\bullet\bullet} = (K, d_1, d_2)$ represents an *anti-commuting* double complex if K is a family $(K^{p,q})$ of objects in $\mathscr{A}$ indexed by $\mathbb{Z} \times \mathbb{Z}$ and $d_1 = (d_1^{p,q}\colon K^{p,q} \to K^{p+1,q})$, $d_2 = (d_2^{p,q}\colon K^{p,q} \to K^{p,q+1})$ are families of maps indexed by $(p,q) \in \mathbb{Z} \times \mathbb{Z}$, called the *horizontal* and *vertical* partial coboundaries respectively, such that

$$d_1 d_1 = 0 \qquad d_2 d_2 = 0, \qquad d_1 d_2 = -d_2 d_1.$$

We set (and please pay attention to the notation, especially the accent on the top left)

$${}'\mathrm{Tot}^n K^{\bullet\bullet} := \bigoplus_{p+q=n} K^{p,q}$$

and define

$$d^n \colon {}'\mathrm{Tot}^n K^{\bullet\bullet} \to {}'\mathrm{Tot}^{n+1} K^{\bullet\bullet}$$

by the formula

$$d^n = \sum_{p+q=n} (d_1^{p,q} + d_2^{p,q})$$

without any sign of the form $(-1)^p$ intervening. It is easy to see, with $d := (d^n)_{n\in\mathbb{Z}}$, that $({}'\mathrm{Tot}^\bullet K^{\bullet\bullet}, d)$ is a complex. We call this complex the total complex associated with the anti-commuting double complex $K^{\bullet\bullet}$.

I will leave the task of defining maps of anti-commuting double complexes to you.

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[1]This is where our assumption that $\mathscr{A}$ has countable direct sums comes into play. Alternately, one can assume that the displayed direct sum for $\mathrm{Tot}^n A^{\bullet\bullet}$ is finite for every $n \in \mathbb{Z}$.

Bounded double complexes. Let $C^{\bullet\bullet}$ be a double complex (standard or anti-commutative). We say it is *bounded on the left* if there is an integer p_0 such that

$$C^{p,q} = 0, \qquad p < p_0.$$

If this happens we sometimes say $C^{\bullet\bullet}$ is *bounded on the left by* p_0. Similarly $C^{\bullet\bullet}$ is *bounded below (by* q_0*)* if there exists an integer q_0 such that

$$C^{p,q} = 0 \qquad q < q_0.$$

I leave to you the fun task of defining terms like *bounded on the right* and *bounded above.*

Note that if $C^{\bullet\bullet}$ is bounded on the left and below (resp. above and to the right) it lives in a translate of the first quadrant (resp. third quadrant) and as such the direct sum

$$\bigoplus_{p+q=n} C^{p,q}$$

is actually a finite sum[2] for each n. So in such instances, one can define $\mathrm{Tot}^n C^{\bullet\bullet}$ or $'\mathrm{Tot}^n C^{\bullet\bullet}$ (as the case may be) without insisting that $\mathscr{A}$ have countable direct sums. In fact we will largely be dealing with such situations.

Tensor product of complexes and the Eilenberg-Zilber map. Suppose A is a ring, $M^\bullet$ a complex of right A-modules, and $N^\bullet$ a complex of left A-modules. One has a double complex of abelian groups $C^{\bullet\bullet}$ given by $C^{pq} = M^p \otimes_A N^q$ with obvious horizontal and vertical coboundary maps. In a somewhat confusing convention, the total complex $\mathrm{Tot}(C^{\bullet\bullet})$ is denoted $M^\bullet \otimes_A N^\bullet$. It should be denoted $\mathrm{Tot}(M^\bullet \otimes_A N^\bullet)$, and some authors of late do do so. We will stick with the old convention for now. For the record

$$M^\bullet \otimes_A N^\bullet := \mathrm{Tot}(C^{\bullet\bullet}).$$

Suppose now that $x \in h^i(M^\bullet)$ and $y \in H^j(N^\bullet)$ and two elements. Let $\xi \in M^i$ and $\zeta \in N^j$ be cocycles representing these cohomology classes. It is easy to see that $\xi \otimes \zeta \in M^i \otimes_A N^j$ is an $(i+j)$-cocycle in $M^\bullet \otimes_A N^\bullet$. Moreover, one checks easily, the cohomology class of the cocycle $\xi \otimes \zeta$ depends only on x and y and not on the representatives ξ and ζ. I leave it to you work this out. The upshot is that we have a map of abelian groups:

$$H^i(M^\bullet) \otimes_A H^j(N^\bullet) \longrightarrow H^{i+j}(M^\bullet \otimes_A N^\bullet)$$

with $x \otimes y$ mapping to the cohomology class of $\xi \otimes \zeta$ where ξ and ζ are as above. We therefore have (for each $n \in \mathbb{Z}$), a map, the so called *Eilenberg-Zilber map*:

$$\mathcal{EZ}_n = \mathcal{EZ}_n(M^\bullet, N^\bullet)\colon \bigoplus_{i+j=n} H^i(M^\bullet) \otimes_A H^j(N^\bullet) \longrightarrow H^n(M^\bullet \otimes_A N^\bullet).$$

It is straightforward to see that $\mathcal{EZ}_n(M^\bullet, N^\bullet)$ is functorial in $M^\bullet$ and in $N^\bullet$, i.e. it is *bifunctorial* in $(M^\bullet, N^\bullet)$.

$'\mathrm{Tot}^n(M^{\bullet\bullet})$

[2]Draw a picture with such quadrant translates, and look at the intersection of such quadrant translates with lines having slope -1.