Some homological algebra

Mapping Cones:

Let A be a commutative noetherian ring.

This hypothesis is only needed sporadically.

Recall that a map of complexes (i.e., a chain map) of A-modules

$$\varphi^\bullet : M^\bullet \longrightarrow N^\bullet$$

is a quasi-isomorphism (i.e., $H^j(\varphi^\bullet)$ is an isomorphism for all $j \in \mathbb{Z}$) if and only if the mapping cone $C^\bullet_\varphi$ is exact where

$$C^n_\varphi = M^{n+1} \oplus N^n$$

and

$$\partial^n_{C_\varphi} = \begin{pmatrix} -\partial^{n+1}_M & 0 \\ \varphi^{n+1} & \partial^n_N \end{pmatrix}$$

Check:

$$\begin{pmatrix} -\partial^{n+2}_M & 0 \\ \varphi^{n+2} & \partial^{n+1}_N \end{pmatrix}\begin{pmatrix} -\partial^{n+1}_M & 0 \\ \varphi^{n+1} & \partial^n_N \end{pmatrix} = \begin{pmatrix} -\partial^{n+2}_M\partial^{n+1}_M & -\partial^{n+2}_M \cdot 0 + 0 \cdot \partial^n_N \\ \underbrace{-\varphi^{n+2}\partial^{n+1}_M + \partial^{n+1}_N\varphi^{n+1}} & \partial^{n+1}_N\partial^n_N \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

zero because $\varphi^\bullet$ is map of cplxes.

This follows from the exact sequence of complexes

$$0 \longrightarrow N^\bullet \longrightarrow C^\bullet_\varphi \longrightarrow M^\bullet[1] \longrightarrow 0$$

and the fact that in the resulting long exact sequence of homologies, the connecting maps are $H^n(\varphi)$.

Bounded above flat complexes:

Lemma 1: Suppose

$$P^\bullet : \quad \cdots \longrightarrow P^{N-1} \longrightarrow P^N \longrightarrow 0$$

is a bounded above exact sequence of flat modules. Then for any A-module M, $P^\bullet \otimes_A M$ is exact.

Proof: WLOG, can assume $N = 0$. Then $P^\bullet$ is a flat resolution of O, and hence $H^{-i}(P^\bullet \otimes M) = \mathrm{Tor}_i^A(M, D) = 0$ $\forall\, i \in \mathbb{Z}$. q.e.d.

Note: A deliberately different proof.

Lemma 2: Let

$$\varphi^\bullet : P^\bullet \longrightarrow Q^\bullet$$

be a quasi-isomorphism between two bounded above flat complexes. Then $\varphi^\bullet \otimes M : P^\bullet \otimes_A M \longrightarrow Q^\bullet \otimes_A M$ is a quasi-isomorphism for every A-module M.

Proof:

$C^\bullet_\varphi$ is an exact bounded above complex of flat A-modules. From Lemma 1, $C^\bullet_\varphi \otimes_A M$ is exact. But clearly $C^\bullet_\varphi \otimes_A M = C^\bullet_{\varphi \otimes M}$. Hence $\varphi^\bullet \otimes M$ is a quasi-isomorphism. q.e.d

Lemma 3: Let $C^\bullet$ be a bdd above complex of A-modules s.t. $H^n(C^\bullet)$ is finitely generated $\forall\, n \in \mathbb{Z}$. Then $C^\bullet$ is quasi-isomorphic to a bounded above complex $D^\bullet$ of finitely generated flat A-modules.

Proof: Standard. Hartshorne also has it. I have no improvement on the usual proof.

Lemma 4: Let A be a local ring with maximal ideal ideal $\mathfrak{m}$ and residue field k. Let $C^\bullet$ be the complex

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow 0$$

consisting of finite A-modules with C^1, C^2 free over A.

(1) Suppose the natural map

$$H^1(C^\bullet)\otimes_A k \longrightarrow H^1(C^\bullet \otimes_A k)$$

is surjective. Then

(a) $Z^1(C^\bullet)$ is a direct summand of C^1 and $B^2(C^\bullet)$ is a direct summand of C^2.

(b) For any A-module M, the natural map

$$H^1(C^\bullet)\otimes_A M \longrightarrow H^1(C^\bullet \otimes_A M)$$

is an isomorphism

(2) If $H^1(C^\bullet \otimes_A k) = 0$ then $H^1(C^\bullet) = 0$.

$\boxed{\text{See note on Nitsure's lemma for a proof.}}$